Steklov–Lyapunov type systems *

A. Bolsinov and Yu. Fedorov
Department of Mathematics and Mechanics
Moscow Lomonosov University, Moscow, 119 899, Russia
bolsinov@mech.math.msu.su, fedorov@mech.math.msu.su

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Abstract

In this paper we describe integrable generalizations of the classical Steklov–Lyapunov systems, which are defined on a certain product $so(m) \times so(m)$, as well as the structure of rank r coadjoint orbits in $so(m) \times so(m)$. We show that the restriction of these systems onto some subvarieties of the orbits written in new matrix variables admits a new $r \times r$ matrix Lax representation in a generalized Gaudin form with a rational spectral parameter.

In the case of rank 2 orbits a corresponding 2×2 Lax pair for the reduced systems enables us to perform a separation of variables.

1 Introduction. Gaudin magnets and the hierarchy of the Steklov–Lyapunov systems.

Many finite-dimensional integrable systems, as well as finite-gap reductions of some integrable PDE's, can be regarded as Hamiltonian flows on finite-dimensional coadjoint orbits of the loop algebra $\tilde{gl}(r)$ described by $r \times r$ Lax equations with a spectral parameter $\lambda \in \mathbb{C}$,

$$\dot{L}(\lambda) = [\, L(\lambda), \mathcal{M}(\lambda)\,], \qquad L = Y + \sum_{i=1}^{n} \frac{\mathcal{N}_i}{\lambda - a_i}, \quad L, \mathcal{M} \in gl(r), \tag{1.1}$$

where $\mathcal{N}_i$ are $r \times r$ matrix variables, $Y \in gl(r)$ is a constant matrix and $a_1, \dots, a_n$ are arbitrary distinct constants (see [1, 2]). In particular, $L(\lambda)$ can be taken in form

$$L(\lambda) = Y + G^T (\lambda \mathbf{I}_n - A)^{-1} F \tag{1.2}$$

where $\mathbf{I}_n$ is the $n \times n$ unit matrix and G, F are $n \times r$ matrices of rank r. Integrable systems described by the corresponding Lax equations are usually referred to as *Gaudin magnets* ([8]) .

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As shown in [1], such systems naturally arise in connection with so called rank r perturbations of the constant matrix $A = \mathrm{diag}(a_1, \dots, a_n)$, namely

$$A \to \mathcal{L}(\nu) = A + F(Y - \nu I_r)^{-1} G^T, \qquad \nu \in \mathbb{C},$$

where now $\mathbf{I}_r$ is the $r \times r$ unit matrix. The matrices $L(\lambda), \mathcal{L}(\nu)$ are *dual* in the sense that their spectral curves are birationally equivalent and the parameter ν plays the role of the eigenvalue parameter for $L(\lambda)$. The characteristic polynomials of the dual Lax matrices are related by the Weinstein–Aronzjan formula (see [1, 11])

$$\begin{aligned} &\det(\lambda \mathbf{I}_n - A) \det(Y + G^T(\lambda \mathbf{I}_n - A)^{-1} F - \nu \mathbf{I}_r) \\ &\quad = \det(\nu \mathbf{I}_r - Y) \det(A + F(Y - \nu I_r)^{-1} G^T - \lambda \mathbf{I}_n) \end{aligned} \tag{1.3}$$

On the other hand, there exists a series of integrable systems which are known to admit a Lax pair with an elliptic spectral parameter only. The examples that we consider here are integrable cases of the classical Kirchhoff equations found by Steklov and Lyapunov ([16, 13]).

Recall that the Kirchhoff equations on the Lie coalgebra $e^*(3) = (K, p)$, $K = (K_1, K_2, K_3)^T$, $p = (p_1, p_2, p_3)^T$ are Hamiltonian with respect to the standard Lie–Poisson bracket

$$\{K_\alpha, K_\beta\} = \varepsilon_{\alpha\beta\gamma} K_\gamma, \quad \{K_\alpha, p_\beta\} = \varepsilon_{\alpha\beta\gamma} p_\gamma, \quad \{p_\alpha, p_\beta\} = 0, \\ (\alpha, \beta, \gamma) = (1, 2, 3),$$

Here (K, p), (p, p) are Casimir functions of the bracket. The Steklov and Lyapunov systems are described respectively by the Hamiltonians

$$\begin{aligned} H_S &= \frac{1}{2} \sum_{\alpha=1}^{3} \left(b_\alpha K_\alpha^2 + 2\nu b_\beta b_\gamma K_\alpha p_\alpha + \nu^2 b_\alpha (b_\beta - b_\gamma)^2 p_\alpha^2 \right), \\ H_L &= \frac{1}{2} \sum_{\alpha=1}^{3} \left(K_\alpha^2 - 2\nu b_\alpha K_\alpha p_\alpha + \nu^2 (b_\beta - b_\gamma)^2 p_\alpha^2 \right), \\ &\quad b_1, b_2, b_3, \ \ \nu = \mathrm{const}, \quad (\alpha, \beta, \gamma) = (1, 2, 3), \end{aligned} \tag{1.4}$$

where ν is an arbitrary parameter.

It can be checked that $\{H_S, H_L\} = 0$ with respect to the above Poisson bracket on $e^*(3)$, which implies the integrability of the Steklov and Lyapunov systems.

These systems were explicitly integrated by Kötter [12], who used the change of variables $(K, p) \to (z, p)$:

$$z_\alpha = K_\alpha - \frac{\nu}{2}(b_\beta + b_\gamma) p_\alpha, \qquad \alpha = 1, 2, 3, \quad (\alpha, \beta, \gamma) = (1, 2, 3) \tag{1.5}$$

and actually represented the equations of motion in a Lax form

$$\begin{aligned} \dot{L}(s) &= [\, L(s), A(s) \,], \qquad L(s), A(s) \in so(3), \quad s \in \mathbb{C}, \\ L_{\alpha\beta}(s) &= \varepsilon_{\alpha\beta\gamma} \sqrt{s - b_\gamma}\, (z_\gamma + s p_\gamma), \end{aligned} \tag{1.6}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor and the matrix $A(s)$ depends on the Hamiltonian of the problem.

The roots in (1.6) are single-valued functions on the elliptic curve $\widehat{\Sigma}$, the 4-sheeted unramified covering of the plane curve $\Sigma = \{w^2 = (s-b_1)(s-b_2)(s-b_3)\}$, which is obtained by doubling of both periods of Σ. This implies that the Lax pair is elliptic. Equivalent $su(2)$ matrix Lax pairs, where the roots are replaced by elliptic functions on Σ are indicated in [3].

According to [5, 9], the Steklov–Lyapunov systems admit multidimensional integrable generalizations defined not on the coalgebra $e^*(n)$, as one might expect, but on a product $so(m) \times so(m)$ with matrix variables $Z, P \in so^*(m)$. The generalized systems admit a Lax pair with a hyperelliptic spectral parameter.

Contents of the paper. In Section 2 we briefly describe m-dimensional Hamiltonian Steklov–Lyapunov systems, the Poisson structure on $so(m) \times so(m)$, and the structure of generic and rank r coadjoint orbits $\mathcal{S}^r_{c,d}$ in $so(m) \times so(m)$, which are characterized by values c, d of the corresponding Casimir functions.

Section 3 shows that the restriction of m-dimensional Steklov–Lyapunov systems onto certain invariant subvarieties $\mathcal{F}^r_{c,d}$ of $\mathcal{S}^r_{c,d}$ admits $r \times r$ matrix Lax representation in a generalized Gaudin form. Namely, the $r \times m$ matrices F, G in (1.2) became linear functions of the spectral parameter λ:

$$G = (\mathcal{X}\,,\ -\mathcal{Y} - \lambda\mathcal{V}), \quad F = (\mathcal{Y} + \lambda\mathcal{V}\,,\ \mathcal{X}),$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{V}$ are $(r/2) \times m$ matrices related to the variables $(Z, P) \in so(m) \times so(m)$ as follows

$$\forall s \in \mathbb{R}, \quad Z + sP = \mathcal{X}^T(\mathcal{Y} + s\mathcal{V}) - (\mathcal{Y} + s\mathcal{V})^T\mathcal{X}\,,$$

so that the corresponding $r \times r$ Lax matrix $L(\lambda)$ obtains a linear part in the spectral parameter:

$$\begin{aligned} L(\lambda) &= \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\,\mathcal{V}]^T & \mathcal{X}(\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \\ -(\mathcal{Y} + \lambda\mathcal{V})(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\mathcal{V}]^T & -[\mathcal{Y} + \lambda\mathcal{V}](\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \end{pmatrix} \\ &= L_1\lambda + L_0 + (\mathcal{X}\ \ -\mathcal{Z})^T(\lambda\mathbf{I}_n - B)^{-1}(\mathcal{Z}\ \mathcal{X})\,, \end{aligned} \tag{1.7}$$

where $B = \operatorname{diag}(b_1, \dots, b_m)$, $\mathcal{Z} = \mathcal{Y} + B\mathcal{V}$, and L_1, L_0 are certain off-diagonal matrices. This Lax matrix leads to a new rational Lax pair for Steklov–Lyapunov systems on $\mathcal{F}^r_{c,d}$. Note that, apparently, in this case the Weinstein–Aronzjan formula (1.3) is not applicable and the dual Lax matrix of $L(\lambda)$ may not exist.

In Section 4 we consider in detail the motion on rank 2 orbits $\mathcal{S}^2_{c,d}$ and show that it allows a special version of the Marsden–Weinstein reduction onto certain symplectic $2(m-1)$-dimensional manifolds $\mathcal{O}^2_{c,d}$. The latter are foliated with $(m-1)$-dimensional Jacobians of hyperelliptic curves, and the reduced systems are just standard algebraic completely integrable Jacobi–Mumford systems (see, e.g., [2, 17]).

In Section 5 we perform a separation of variables for the reduced systems by indicating the Abel–Jacobi quadratures in terms of certain coordinates on $\mathcal{O}^2_{c,d}$, which are Darboux coordinates with respect to the original Lie–Poisson structure on $so(m) \times so(m)$.

In the classical case $m = 3$, the coadjoint orbits $\mathcal{S}^2_{c,d}$ are just coverings of $\mathcal{O}^2_{c,d}$, and the above coordinates, as separating variables, were first introduced by F. Kötter in his short paper [12] without discussing their symplectic nature.

In Conclusion we summarise our results and outline further integrable generalizations of multidimensional Steklov–Lyapunov systems and their Lax representations in a generalized Gaudin form.

2 Steklov–Lyapunov systems on generic and special rank r coadjoint orbits in $so(m) \times so(m)$

Following [5, 9], multidimensional Steklov–Lyapunov systems are defined on a product $so(m) \times so(m)$ with matrix variables $Z, P \in so(m)$, which is endowed with the following Poisson bracket

$$
\begin{aligned}
\{f, h\}_1 &= \langle Z, [\, d_Z f, d_Z h\,]\rangle + \langle P, [\, d_Z f, d_P h\,] + [\, d_P f, d_Z h\,]\rangle \\
&\quad - \langle P, (d_Z f\, B\, d_Z h - d_Z h\, B\, d_Z f)\rangle\,, \qquad (2.1)\\
B &= \operatorname{diag}(b_1, \dots, b_m),\\
(d_Z f)_{ij} &= \partial f/\partial Z_{ij}\,, \quad (d_P f)_{ij} = \partial f/\partial P_{ij}\,, \qquad i, j = 1, \dots, m
\end{aligned}
$$

where $b_1, \dots, b_m$ are arbitrary distinct constants, and for $X, Y \in so(m)$, $\langle X, Y\rangle = -\frac{1}{2}\operatorname{tr}(XY)$. This implies that equations of motion can be written in the Hamiltonian form

$$
\begin{aligned}
\dot Z &= \left[Z, \frac{\partial \mathcal{H}}{\partial Z}\right] + B\frac{\partial \mathcal{H}}{\partial Z}P - P\frac{\partial \mathcal{H}}{\partial Z}B + \left[P, \frac{\partial \mathcal{H}}{\partial P}\right],\\
\dot P &= \left[P, \frac{\partial \mathcal{H}}{\partial Z}\right].
\end{aligned} \qquad (2.2)
$$

The bracket $\{f, h\}_1$ has exactly $2[m/2]$ independent Casimir functions

$$
\begin{aligned}
\mathcal{P}_k &= -\operatorname{tr}(P^k)\,,\\
\mathcal{Q}_k &= \operatorname{tr}(ZP^{k-1} + P^k B)\,,
\end{aligned} \qquad k = 2, 4, \dots, 2[m/2] \qquad (2.3)
$$

(here and below, the parenthesis [ ] with one argument denotes the integer part of the number).

The multidimensional integrable analogs of the Lyapunov and Steklov systems are described by the following quadratic Hamiltonians that generalize (1.4),

$$
\begin{aligned}
\mathcal{H}_L &= \langle Z, Z\rangle + 2\langle Z, (BP + PB)\rangle + \langle P, (B^2P + BPB + PB^2)\rangle\,,\\
\mathcal{H}_S &= \langle Z, BZ + ZB\rangle + 2\langle Z, \{P, B^2\}\rangle + \langle P, \{P, B^3\}\rangle - \mathbf{tr}\, B\, \mathcal{H}_L. \qquad (2.4)
\end{aligned}
$$

Here and below the bracket $\{X^l, Y^r\}$ (without an index) denotes a homogeneous symmetric matrix polynomial in X and Y of degrees s and r respectively, for example: $\{X, Y^0\} = X$, $\{X, Y\} = XY + YX$, $\{X, Y^2\} = XY^2 + YXY + Y^2X$, etc.

The corresponding flows admit the following Lax pairs, which generalize (1.6),

$$\dot{L}(s) = [\,L(s), A(s)\,]\,, \qquad L(s)\,, A(s) \in so(m)\,, \quad s \in \mathbb{C}\,, \tag{2.5}$$

$$L(s)_{ij} = \frac{\sqrt{\Phi(s)}}{\sqrt{(s-b_i)(s-b_j)}}\,(Z+sP)_{ij}\,, \qquad i,j=1,\dots,m\,, \tag{2.6}$$

$$\Phi(s) = (s-b_1)\cdots(s-b_m)\,, \qquad b_1,\dots,b_m = \text{const},$$

where the roots $w_{ij} = \sqrt{(s-b_i)(s-b_j)}$ are assumed to satisfy the relations $w_{ik}w_{kj} = (s-b_k)w_{ij}$. Under this condition, the roots, as well as $\sqrt{\Phi(s)}$, are single-valued functions on an unramified covering of the hyperelliptic curve $\Sigma = \{w^2 = \Phi(s)\}$. In this connection the Lax pair (2.5) is referred to as *hyperelliptic*.

To obtain the generalized Lyapunov and Steklov systems, in (2.5) we put

$$A(s)_{ij} = -\frac{1}{s}\sqrt{(s-b_i)(s-b_j)}\,P_{ij}, \qquad \text{and, respectively,}$$

$$A(s)_{ij} = \sqrt{(s-b_i)(s-b_j)}\,(sP_{ij} + Z_{ij} + (\mathbf{tr}\,B\, - b_i - b_j)P_{ij})\,.$$

Moreover, as shown in [5, 9], there exists a hierarchy of "higher" Steklov–Lyapunov systems. In particular, putting in (2.5)

$$A = A_{1,\rho}(s) = -S\tilde{A}_{1,\rho}(s)S, \qquad S = \operatorname{diag}(\sqrt{s-b_1},\dots,\sqrt{s-b_m}), \quad \rho = 0,1,2,\dots,$$

$$\tilde{A}_{1,\rho}(s) = s^\rho P + s^{\rho-1}\{B,P\} + \cdots + \{B^\rho, P\} + s^{\rho-1}Z + \cdots + \{B^{\rho-1}, Z\},$$

we obtain the following subhierarchy of systems with quadratic right hand sides

$$\begin{aligned} \dot{Z} &= [Z, \{Z, B^\rho\}] + Z\{P, B^\rho\}B - B\{P, B^\rho\}Z\,, \\ \dot{P} &= [P, \{P, B^{\rho+1}\}] + [P, \{Z, B^\rho\}], \qquad \rho \in \{0, \mathbb{N}\}. \end{aligned} \tag{2.7}$$

The matrix $A_{1,0}$ coincides with the above operator defining the multidimensional generalization of the Lyapunov system.

Following [5], apart from the bracket $\{\,,\,\}_1$, on $so(m)\times so(m)$ there is another Poisson bracket $\{\,,\,\}_0$, such that $\{\,,\,\}_1$, $\{\,,\,\}_0$ form a pencil of consistent (or compatible) Poisson brackets. The coefficients of the spectral curve provide a complete set of first integrals in involution with respect to all the brackets of the pencil, which proves the Liouville integrability of all the systems of the hierarchy.

The coadjoint action. Under the change of matrix variables

$$(Z,P) \to (M,P)\,: \quad M = Z + \frac{1}{2}(BP+PB), \tag{2.8}$$

which is actually a generalization of Kötter's substitution (1.5), the bracket $\{f,h\}_1$ becomes precisely the Lie–Poisson bracket of the semi-direct Lie algebra product $so(m)\times_s so(m)$ specified by the commutator

$$\text{for } \ (X,Y) \in so(m)\times_s so(m)\,,$$

$$[(X_1,Y_1),(X_2,Y_2)] = ([X_1,X_2],\,[X_1,Y_2]-[X_2,Y_1]). \tag{2.9}$$

Indeed, for (M,P) in the dual space to $so(m)\times_s so(m)$, we introduce the natural pairing

$$\langle (M,P),(X,Y)\rangle = \langle M,X\rangle + \langle P,Y\rangle.$$

Then, by the definition of a Lie–Poisson bracket and in view of (2.9),

$$\{f(M,P),h(M,P)\}_1 = \langle M,[\,d_M f, d_M h]\rangle + \langle P,[\,d_M f, d_P h] + [\,d_P f, d_M h]\rangle\,,$$

which transforms to (2.1) under the substitution (2.8).

In the classical case $m=3$, in the vector variables K,p such that

$$Z_{ij} = \varepsilon_{ijk}(K_k - \frac{1}{2}(b_i+b_j)p_k), \quad P_{ij} = \varepsilon_{ijk}p_k, \tag{2.10}$$

the bracket $\{f,h\}_1$ is just the Lie–Poisson bracket on $e^*(3)$. —medskip

The space $so(m)\times_s so(m)$ is the Lie algebra of the semidirect group product $G = SO(n)\times_{Ad} so(n)$. The elements of this group are pairs of the form (R,ϱ), $R\in SO(n)$, $\varrho\in so(n)$, the multiplication is given by the following natural formula:

$$(R_1,\varrho_1)\cdot(R_2,\varrho_2) = (R_1R_2, R_1\varrho_2R_1^{-1} + \varrho_1),$$

and the inverse of (R,ϱ) is $(R^{-1}, -R^{-1}\varrho R)$.

Using (2.9), one can verify that the adjoint action of G on its Lie algebra has the form

$$Ad_{(R,\varrho)}(X,Y) = (RXR^{-1}, RYR^{-1} - [RXR^{-1},\varrho]),$$

and

$$Ad^{-1}_{(R,\varrho)}(X,Y) = Ad_{(R^{-1},-R^{-1}\varrho R)}(X,Y) = (R^{-1}XR, R^{-1}YR + R^{-1}[X,\varrho]R).$$

To derive the explicit formula for the coadjoint action, we use the definition

$$\langle Ad^{-1}_{(R,\varrho)}(X,Y),(M,P)\rangle = \langle (X,Y), Ad^*_{(R,\varrho)}(M,P)\rangle.$$

Then we have

$$\begin{aligned}
&\langle (X,Y), Ad^*_{(R,\varrho)}(M,P)\rangle = \langle Ad^{-1}_{(R,\varrho)}(a,b),(M,P)\rangle = \\
&\langle (R^{-1}XR, R^{-1}YR + R^{-1}[X,\varrho]R),(M,P)\rangle = \\
&\langle R^{-1}XR, M\rangle + \langle (R^{-1}YR + R^{-1}[X,\varrho]R), P\rangle = \\
&\langle X, RMR^{-1}\rangle + \langle Y, RPR^{-1}\rangle + \langle X, [\varrho, RPR^{-1}]\rangle = \\
&\langle (X,Y), (RMR^{-1} + [\varrho, RPR^{-1}], RPR^{-1})\rangle.
\end{aligned}$$

Hence

$$Ad^*_{(R,\varrho)}(M,P) = (RMR^{-1} + [\varrho, RPR^{-1}], RPR^{-1}) \tag{2.11}$$

Using this formula we can describe the Casimir functions of the Lie-Poisson bracket (2.1) as invariants of the above coadjoint action. Namely, let $f(P)$ be an arbitrary Ad-invariant function on the Lie algebra $so(m)$. Then, since G acts on the second component P by conjugations, this function is also Ad^*-invariant in the sense of the algebra $so(m) \times_s so(m)$. Moreover the function

$$h(M,P) = \lim_{t\to 0} \frac{f(P+tM) - f(P)}{t} = \langle df(P)M \rangle,$$

where $df(P) \in so(m)$ is the differential of f at the point P, is also Ad^*-invariant Indeed,

$$\begin{aligned}
&h(XMX^{-1} + [\varrho, XPX^{-1}], XPX^{-1}) \\
&= \lim_{t\to 0} \frac{f(X(P + tM + t[X^{-1}\varrho X, P])X^{-1}) - f(XPX^{-1})}{t} \\
&= \lim_{t\to 0} \frac{f(P + tM + t[X^{-1}\varrho X, P]) - f(P)}{t} \\
&= \operatorname{tr} df(P)(M + [X^{-1}\varrho X, P]) \\
&= \operatorname{tr} df(P)M + \operatorname{tr} df(P)[X^{-1}\varrho X, P] \\
&= \operatorname{tr} df(P)M + \operatorname{tr}[P, df(P)]X^{-1}\varrho X = \operatorname{tr} df(P)M = h(M,P).
\end{aligned}$$

(Here we use the fact that $[P, df(P)] = 0$ for any Ad-invariant function $f(P)$). In particular, the functions

$$\mathcal{P}_k = \operatorname{tr} P^k, \quad \mathcal{Q}_k = \operatorname{tr}(MP^{k-1}), \quad k = 2, 4, \ldots, 2[n/2]$$

are independent Ad^*-invariants (if n is even then the two last functions can be replaced by $\mathrm{Pf}(P)$ and $\operatorname{tr}(d\,\mathrm{Pf}(P))M)$. Under the substitution (2.8) they transform to the Casimir functions (2.3).

Although the matrix variables M, P are more convenient than Z, P from the point of view of algebraic and Hamiltonian description, for our future purposes we shall continue using both sets of variables.

First integrals and generic coadjoint orbits. The characteristic polynomial of the Lax matrix (2.6) has the form

$$|L(s) - wI| = w^m + \sum_k w^{m-k}\, \Phi^{k/2-1}(s)\, \widetilde{\mathcal{I}}_k(s, Z, P), \tag{2.12}$$

$$k = 2, \ldots, 2[m/2] \quad (k \text{ is even}),$$

$$\widetilde{\mathcal{I}}_k(s, Z, P) = \sum_I \frac{\Phi(s)}{(s - b_{i_1}) \ldots (s - b_{i_k})}\, |Z + sP|_I^I = \sum_{\mu=0}^{m} s^\mu H_{k\mu}(Z, P), \tag{2.13}$$

where $|Z + sP|_I^I$ denotes the k-order diagonal minor corresponding to the multi-index $I = \{i_1 \ldots i_k\}$, $i_1 < \cdots < i_k$, which ranges over the set of all such indices. In particular, the two major coefficients

$$
\begin{aligned}
H_{km} &= \sum_I |P|_I^I \,, \\
H_{k,m-1} &= \sum_I (b_{i_1} + \cdots + b_{i_k}) |P|_I^I - (\operatorname{tr} B)\, H_{km}(P) + \operatorname*{\mathit{Res}}_{\varkappa=0} \sum_I \left| \varkappa^{-1} Z + P \right|_I^I \\
&\equiv \operatorname*{\mathit{Res}}_{\varkappa=0} \sum_I \left| \varkappa^{-1} M + P \right|_I^I - (\operatorname{tr} B)\, H_{km}(P)
\end{aligned}
\tag{2.14}
$$

are annihilators of the bracket (2.1), and they are linear combinations of the Casimir functions (2.3).

The family of quadratic integrals has the form

$$
\begin{aligned}
\widetilde{\mathcal{I}}_2(\lambda, Z, P) &= \sum_{i<j}^m \frac{\Phi(\lambda)}{(\lambda - b_i)(\lambda - b_j)} (Z_{ij} + \lambda P_{ij})^2 \\
&= \lambda^m \langle P, P \rangle + H_{2,m-1}(Z,P) \lambda^{m-1} + H_{2,m-2}(Z,P) \lambda^{m-2} \\
&\quad + H_{2,m-3}(Z,P) \lambda^{m-3} + \cdots + H_{2,0}(Z,P),
\end{aligned}
\tag{2.15}
$$

where

$$
\begin{aligned}
H_{2,m-1} &= \mathfrak{h}_{m-1} - \Delta_1 \mathfrak{h}_m, \\
H_{2,m-2} &= \mathfrak{h}_{m-2} - \Delta_1 \mathfrak{h}_{m-1} + \Delta_2 \mathfrak{h}_m, \\
H_{2,m-3} &= \mathfrak{h}_{m-3} - \Delta_1 \mathfrak{h}_{m-2} + \Delta_2 \mathfrak{h}_{m-1} - \Delta_3 \mathfrak{h}_m, \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \\
H_{2,0} &= \sum_{s=0}^m (-1)^s \Delta_s \mathfrak{h}_s = \det B \langle Z, B^{-1} Z B^{-1} \rangle.
\end{aligned}
\tag{2.16}
$$

and where $\mathfrak{h}_s(Z,P)$ are integrals in a "canonical" form,

$$
\begin{gathered}
\mathfrak{h}_m = \langle P, P \rangle, \quad \mathfrak{h}_{m-1} = 2\langle Z, P \rangle + \langle P, BP + PB \rangle, \\
\mathfrak{h}_{m-2-\varrho} = \langle Z, \{Z, B^\varrho\} \rangle + 2\langle Z, \{P, B^{\varrho+1}\} \rangle + \langle P, \{P, B^{\rho+2}\} \rangle \,, \\
\rho = 0, 1, \ldots, m-2,
\end{gathered}
$$

Δ_s being elementary symmetric functions of $b_1, \ldots, b_m$ of degree s, e.g., $\Delta_0 = 1$, $\Delta_1 = b_1 + \cdots + b_m$, $\Delta_2 = b_1 b_2 + \cdots + b_{m-1} b_m$, etc.

Notice that $\langle P, P \rangle$ and $H_{2,m-1}$ are quadratic Casimir functions, whereas (up to adding such functions) $H_{2,m-2}$ and $H_{2,m-3}$ coincide with the Lyapunov and Steklov Hamiltonians in (2.4) respectively.

As shown in [5, 9], for odd dimension m, the polynomials

$$
H_{k\nu}(Z,P), \qquad k = 2, 4, \cdots < m, \quad \nu = 0, 1, \ldots, m-2
$$

form a complete involutive set of $(m-1)[m/2]$ independent first integrals of the systems. The same holds for even dimension m with the only exception: the polynomial $\widetilde{\mathcal{I}}_m(s)$ is the full square of a polynomial $\mathcal{I}'_m(s)$ of degree $m/2$ in Z, P, which is the Pfaffian of $L(s)$. The coefficients of $\mathcal{I}'_m(s)$ are independent of each other and of the integrals $H_{k\nu}$ with $k = 2, \ldots, m-2$. The two major coefficients of $\mathcal{I}'_m(s)$ are again annulators of the bracket (2.1). Thus, for even m, we have a complete set of $m(m-2)/2$ independent first integrals in involution (see [5, 9]).

On the other hand, a generic symplectic leave of the Poisson bracket (2.1),

$$\mathcal{S}_{c,d} = \left\{ Z, P \mid H_{km}(P) = c_k,\ H_{k,m-1}(Z,P) = d_k, \qquad k = 2, 4, \ldots, 2[m/2] \right\}$$

c_k, d_k being generic constants, has dimension $m(m-1) - 2[m/2]$. As follows from above, for even and odd m, this is twice the number of the involutive integrals $H_{k,m-2}(P,Z), \ldots, H_{k,0}(Z)$. As a result, the dimension of *generic* invariant tori of the Steklov–Lyapunov systems equals $m(m-1)/2 - [m/2]$.

Simple rank r orbits. Apart from generic coadjoint orbits there exists a hierarchy of lower-dimensional orbits corresponding to certain special values of c_k, d_k. We now describe the structure of such orbits in the coalgebra. As follows from the formula (2.11), on each orbit the rank of P is constant. (Notice that, the rank of M is not !) We assume rank $P = r \leq m$. Then, by the Ad^*-action this matrix can be reduced to the following canonical form

$$P^* = \begin{pmatrix} P_1 & & & \\ & \ddots & & \\ & & P_k & \\ & & & P_0 \end{pmatrix}$$

where $\dim P_1 + \cdots + \dim P_k = r$, P_0 is the zero matrix and for $i \neq 0$,

$$P_i = \begin{pmatrix} 0 & p_i & & & \\ -p_i & 0 & & & \\ & & \ddots & & \\ & & & 0 & p_i \\ & & & -p_i & 0 \end{pmatrix}, \qquad 0 < p_1 < \cdots < p_k.$$

Here and below the empty enties are zero entries. One can check that the annihilator of P^*,

$$\mathrm{Ann}(P^*) = \{A \in so(m) \mid [A, P^*] = 0\}$$

consists of the matrices of the form

$$A = \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_k & \\ & & & A_0 \end{pmatrix}$$

where A_0 is an arbitrary matrix of $so(m-r)$, each block A_i has the same dimension as P_i and the following block structure

$$A_i = \begin{pmatrix} C_{11} & \cdots & C_{1\,l_i} \\ \vdots & \ddots & \vdots \\ -C_{1\,l_i}^T & \cdots & C_{l_i\,l_i} \end{pmatrix}, \qquad l_i = \frac{1}{2}\dim P_i, \quad C_{jk} = \begin{pmatrix} c_{jk} & d_{jk} \\ -d_{jk} & c_{jk} \end{pmatrix},$$

with $c_{jj} = 0$. Notice that C_{jk} is the standard matrix representation of a complex number $c_{jk} + \sqrt{-1}d_{jk}$, hence each block A_i can be regarded a complex unitary matrix, i.e., as a matrix of the Lie algebra $u(l_i)$. Thus

$$\mathrm{Ann}(P^*) = u(l_1) \oplus u(l_2) \oplus \cdots \oplus u(l_k) \oplus so(m-r), \qquad 2l_1 + \cdots + 2l_k = r.$$

Now we describe the canonical form for M (having fixed the canonical form P^* of P). First divide M into two parts $M = M_1 + M_2$ where $M_1 \in \mathrm{Ann}(P^*)$, $M_2 \in (\mathrm{Ann}(P^*))^\perp$. Notice that M_2 can be always presented as $M_2 = [\varrho, P^*]$ for a certain operator ϱ. Hence, in view of the formula (2.11), the part M_2 can be killed by the coadjoint action. In other words, without loss of generality we may assume that $M = M_1 \in \mathrm{Ann}(P^*)$.

Further, if one acts on (M_1, P^*) by the elements $(R, 0)$, where R belongs to the Lie subgroup corresponding to $\mathrm{Ann}(P^*)$, then P^* does not change and M_1 can be reduced to the canonical block diagonal form in $\mathrm{Ann}(P)$:

$$M^* = \begin{pmatrix} 0 & m_1 & & & & \\ -m_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & m_{r/2} & \\ & & & -m_{r/2} & 0 & \\ & & & & & A_0 \end{pmatrix}$$

Thus the final conclusion is that M and P can be both reduced to the block diagonal form, and M^* admits a natural decomposition

$$M^* = M_1 + \cdots + M_k + M_0, \qquad M_i \in u(l_i), \quad M_0 \in so(m-r).$$

In the sequel we shall study the family of orbits in $so(m) \times so(m)$ passing through (M^*, P^*) with rank $P = r$ and $M_0 = 0$. We shall call them *simple rank r orbits.*

The corresponding matrix $Z^* = M^* + \frac{1}{2}(BP^* + P^*B)$ has the block structure diag (Z_1, Z_0), where $Z_1 \in so(r)$ and Z_0 is the zero $(m-r) \times (m-r)$ matrix. Hence, for any s, rank $|Z^* + sP^*| \le r$. Then, according to expressions (2.13), (2.14), all the higher Casimir functions

$$H_{r+2,m}(P),\ H_{r+2,m-1}(Z,P),\ \ldots,\quad H_{2[m/2],m}(P),\ H_{2[m/2],m-1}(Z,P)$$

equal zero for $(Z,P) = (Z^*, P^*)$ and, therefore, on the whole simple rank r orbits.

Notice that not all of such orbits have the same dimenion and not all of them are separated by values of the rest of the Casimir functions $H_{k,m}(P)$, $H_{k,m-1}(Z,P)$ of order $\leq r$. They do, if we asume that all the nonzero eigenvalues of P are distinct. Such orbits will be refereed to as *generic simple rank r orbits* and denoted as $\mathcal{S}^r_{c,d}$. As follows from above, by an appropriate action $Ad^*_{R,0}$ *any* point of these orbits can be reduced to the form

$$
P^* = \begin{pmatrix} 0 & p_1 & & & & \\ -p_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & p_{r/2} & \\ & & & -p_{r/2} & 0 & \\ & & & & & \mathbf{0} \end{pmatrix}, \quad \hat{Z} = \begin{pmatrix} \mathcal{Z} & \mathcal{U} \\ -\mathcal{U}^T & \mathbf{0} \end{pmatrix}, \tag{2.17}
$$

$$
p_1 < \cdots < p_{r/2},
$$

where $\mathbf{0}$ is zero $(m-r)\times(m-r)$ matrix, $\mathcal{Z} \in so(r)$ and $\mathcal{U}$ is a generic matrix of dimensional $r \times m$.

Proposition 2.1 *1). Generic simple orbits $\mathcal{S}^r_{c,d}$ has dimension $2r(m-1-r/2)$, which is twice the dimension of generic rank r orbits of coadjoint action of $SO(m)$ on $so^*(m) = \{P\}$.*

2). On them the invariant polynomials $\widetilde{\mathcal{I}}_{2r+2}(s), \dots, \widetilde{\mathcal{I}}_{2[m/2]}(s)$ vanish identically.

3). A complete set of nonzero first integrals and Casimir functions are given by the coefficients of the polynomials

$$
\widetilde{\mathcal{I}}_k(s) = \begin{cases} \displaystyle\sum_I \frac{\Phi(s)}{(s-b_{i_1})\dots(s-b_{i_k})} \, |Z+sP|_I^I, & k = 2,\dots,r, \\ \displaystyle\sum_I \frac{\Phi(s)}{(s-b_{i_1})\dots(s-b_{i_k})} \sum_{j=0}^{g-k} |s^j P^j, Z^{k-j}|_I^I & k = r+2,\dots,g, \end{cases} \tag{2.18}
$$

$$
g = \min\{2r, 2[m/2]\}, \quad \Phi(\lambda) = (\lambda - b_1)\cdots(\lambda - b_m),
$$

where k is even, $|s^j P^j, Z^{k-j}|_I^I$ denotes the diagonal minor of order k with $I = \{i_1 \dots i_k\}$ that contains products of j components of P and $k-j$ components of Z.

These polynomials provide $r(m-1-r/2)$ first integrals in involution, which are independent almost everywhere on $\mathcal{S}^r_{c,d}$, and their generic common level varieties are $r(m-1-r/2)$-dimensional tori.

Sketch of a proof of Proposition 2.1.

1). Let us describe the annihilator of a generic canonical pair (M^*, P^*) with respect to the coadjoint action (2.11),

$$
\operatorname{Ann}(M^*, P^*) = \{(X,Y) \in so(m) +_{ad} so(m) \mid ad^*_{(X,Y)}(M^*, P^*) = (0,0)\}.
$$

We obtain the system of linear equations

$$[X, M^*] + [Y, P^*] = 0, \quad [X, P^*] = 0.$$

This implies that $X \in \mathrm{Ann}(P^*)$. On the other hand, M^* also belongs to $\mathrm{Ann}(P^*)$. Therefore $[X, M^*] \in \mathrm{Ann}(P^*)$, whereas $[Y, P^*] \in (\mathrm{Ann}(P^*))^{\perp}$. Thus, the first equation above decomposes into two independent equations

$$[X, M] = 0, \quad [Y, P] = 0.$$

As a result, we get

$$Y \in \mathrm{Ann}(P^*), \quad X \in \mathrm{Ann}_{\mathrm{Ann}(P^*)}(M^*) = \{l \in \mathrm{Ann}(P^*) \mid [l, M^*] = 0.$$

This means that $\mathrm{Ann}(M^*, P^*)$ is the semidirect sum of $\mathrm{Ann}_{\mathrm{Ann}(P^*)}(M^*)$ and $\mathrm{Ann}(P^*)$ (with respect to the adjoint representation). In particular, we have that codimension of the orbit $\mathcal{O}(M^*, P^*)$ passing through the point (M^*, P^*) equals

$$\operatorname{codim} \mathcal{O}(M^*, P^*) = \dim \mathrm{Ann}(M^*, P^*) = \dim \mathrm{Ann}(P^*) + \dim \mathrm{Ann}_{\mathrm{Ann}(P)}(M). \tag{2.19}$$

This is a particular case of the so-called Rais formula that describes the dimension of coadjoint orbits in semidirect sums of Lie algebras (see, e.g., [10]).

In the case when P is of maximal rang $2[m/2]$ and all its eigenvalues are different, the annihilator of P^* in $so(m)$ is the Cartan subalgebra that contains P^*. Since this subalgebra is commutative, we have $\mathrm{Ann}_{\mathrm{Ann}(P^*)}(M^*) = \mathrm{Ann}(P^*)$. Thus $\mathrm{Ann}(M^*, P^*)$ is the sum of two copies of the Cartan subalgebra: $\mathrm{Ann}(M^*, P^*) = \{(a, b) \mid a, b \in \mathrm{Ann}\, P^*\} \subset so(m) +_{ad} so(m)$, and $\operatorname{codim} \mathcal{O}(M^*, P^*) = 2[m/2]$.

In the case of generic simple rank r orbits the $u(l)$-components of $\mathrm{Ann}(P^*)$ become one-dimensional, i.e., $u(1) = \mathbb{R}$ and, consequently, $\mathrm{Ann}(P^*) = \mathbb{R}^{r/2} \oplus so(m - r)$. Next, M^* belongs to the commutative part $\mathbb{R}^{r/2}$ of $\mathrm{Ann}(P^*)$. Therefore $\mathrm{Ann}_{\mathrm{Ann}(P^*)}(M^*) = \mathrm{Ann}(P^*)$ and, according to the formula (2.19)

$$\operatorname{codim} \mathcal{O}(M^*, P^*) = 2 \dim \mathrm{Ann}(P^*) = 2\left(\frac{r}{2} + \frac{(m-r)(m-r-1)}{2}\right).$$

This gives the dimension of $\mathcal{S}^r_{c,d}$ stated by the proposition.

2) Now we evaluate the first integrals given by (2.13) on the matrices (2.17) that represent a generic point on $\mathcal{S}^r_{c,d}$. Note that for any s, rank $|\hat{Z} + sP^*| \leq 2r$, hence

$$\widetilde{\mathcal{I}}_{2r+2}(s, \hat{Z}, P^*) = 0, \quad \ldots, \quad \widetilde{\mathcal{I}}_{2[m/2]}(s, \hat{Z}, P^*) = 0.$$

3) One can show that for $r < k \leq g$, the minors $|\hat{Z} + sP^*|^I_I$ must contain at least $2(k - r)$ nonzero components of $\hat{Z}$, hence the minors have at most degree $2r - k$ in s and in the components of P^*. Finally, for $2 \leq k \leq r$, there are no restrictions on the degree of the polynomials $|\hat{Z} + sP^*|^I_I$, and all their coefficients are generally nonzero. As a result, the integrals given by (2.13) take the form (2.18) on the entire orbit $\mathcal{S}^r_{c,d}$. The latter formula provides $r(m - 1 - r/2)$ nonzero nontrivial integrals, which is the maximal number of independent integrals in involution on the orbit.

The common level of the integrals belongs to an orbit of the coadjoint action corresponing to the second Poisson bracket $\{\,,\,\}_0$ on $so(m)\times so(m)$. As follows from [5], the latter orbits are always compact. Hence, although $\mathcal{S}^r_{c,d}$ is noncompact, the above common level is a compact variety of generic dimension $r(m-1-r/2)$, which implies that the orbit $\mathcal{S}^r_{c,d}$ almost everwhere is foliated with tori of the same dimension. The proposition is proved.

Remark 1. The orbits $\mathcal{S}^r_{c,d}$ contain invariant subvarieties

$$\mathcal{F}^r_{c,d}=\{(Z,P)\in\mathcal{S}^r_{c,d}\mid \forall s\in\mathbb{R}\,,\quad \text{rank}\,|Z+sP|\le r\}$$

on which the higher order invariant polynomials $\widetilde{\mathcal{I}}_{r+2}(s,Z,P),\dots,\widetilde{\mathcal{I}}_g(s,Z,P)$ are identically zero. Then

$$\dim\mathcal{F}^r_{c,d}=\dim\,\mathcal{S}^r_{c,d}-\text{number of the coefficients of } \widetilde{\mathcal{I}}_{r+2}(s),\dots,\widetilde{\mathcal{I}}_g(s) \text{ in } (2.18)\,,$$

which equals $r(3m/2-3/2-r/2)$. Let $\bar\omega_s$ be the 2-form on $\mathbb{R}^m$ with the components $(Z+sP)_{ij}$. Then, for $r<m$ and a fixed s, the components of the r-form $\bar\omega_s^{r/2}$ can be regarded as Plücker coordinates of an r-plane passing through the origin in $\mathbb{R}^m$, whereas the family of such linear spaces parameterized by s is *a pencil* of r-planes $\mathcal{L}$ having a common $r/2$-plane $\mathfrak{P}$, the focus of $\mathcal{L}$.

3 Flows on the matrix triplet variety

Let $\mathcal{W}^r$ be a union of the subvarieties $\mathcal{F}^r_{c,d}$ corresponding to all nonzero Casimir functions given by (2.18). As follows from Proposition 2.1, there are presicely r such functions. Hence

$$\dim\mathcal{W}^r=\dim\mathcal{F}^r_{c,d}+r=\frac{3}{2}mr-\frac{r}{2}-\frac{r^2}{2}.$$

There exist $r/2$ triples of vectors $x^{(l)},y^{(l)},v^{(l)}\in\mathbb{R}^n$, $l=1,\dots,r/2$ such that any point of $\mathcal{W}^r$ can be represented in form

$$\forall s\in\mathbb{C},\quad Z+sP=\sum_{l=1}^{r/2}x^{(l)}\wedge(y^{(l)}+sv^{(l)})\equiv\mathcal{X}^T(\mathcal{Y}+s\mathcal{V})-(\mathcal{Y}+s\mathcal{V})^T\mathcal{X}\,,\quad (3.1)$$

where $\mathcal{X},\mathcal{Y},\mathcal{V}$ are $r/2\times m$ matrices,

$$\mathcal{X}^T=(x^{(1)}\cdots x^{(r/2)})\,,\quad \mathcal{Y}^T=(y^{(1)}\cdots y^{(r/2)})\,,\quad \mathcal{V}^T=(v^{(1)}\cdots v^{(r/2)})\,.$$

(Notice that the linear span of $x^{(1)},\dots,x^{(r/2)}$ gives the above $r/2$-dimensional focus $\mathfrak{P}$ of $\mathcal{L}$.) It is seen that for a generic pair Z,P, such vectors are not unique. In particular, under the transformations

$$y^{(l)}\to y^{(l)}+\tau_l x^{(l)},\quad v^{(l)}\to v^{(l)}+\delta_l x^{(l)},\qquad \text{for any}\quad \tau_l,\delta_l\in\mathbb{R}$$

Z, P remain unchanged. To get rid of the ambiguity, we introduce constraint submanifold

$$
\mathcal{T}^r = \{\mathcal{X}, \mathcal{Y}, \mathcal{V} \,|\, \mathcal{X}\mathcal{X}^T = \mathbf{I}, \quad \mathcal{V}\mathcal{X}^T = 0, \quad \mathcal{X}[\mathcal{Y} + \mathcal{V}B]^T + [\mathcal{Y} + \mathcal{V}B]\mathcal{X}^T = 0\}, \quad (3.2)
$$

which is defined by $\frac{r}{2} + \frac{r^2}{2}$ scalar constraint equations in $\mathbb{R}^{3mr/2}$ and therefore has the same dimension as $\mathcal{W}^r$. (We shall refer to it as the *matrix triplet variety*.) Then a complete preimage of a generic point of $\mathcal{W}^r$ in $\mathcal{T}^r$ is a discrete orbit of the group $\mathfrak{R}$ generated by reflections

$$
(x^{(l)}, y^{(l)}, v^{(l)}) \to (-x^{(l)}, -y^{(l)}, -v^{(l)}), \qquad l = 1, \dots, r/2.
$$

The main observation of this section is that *the restriction of the Steklov–Lyapunov systems on $\mathcal{W}^r$ can be described as dynamical systems on $\mathcal{T}^r$, which admit $r \times r$ matrix Lax pairs with a rational parameter.*

As model systems, we take equations (2.7), which are described by the quadratic Hamiltonians

$$
\frac{1}{2}\mathfrak{h}_{m-2-\rho} = \frac{1}{2}\langle Z, \{Z, B^\rho\}\rangle + \langle Z, \{P, B^{\rho+1}\}\rangle + \frac{1}{2}\langle P, \{P, B^{\rho+2}\}\rangle, \qquad \rho = 0, 1, \dots,
$$

and can be represented in form

$$
\begin{aligned}
\dot{Z} &= [\, Z, \Omega_\rho \,] + P B^{\rho+1} Z - Z B^{\rho+1} P \,, \\
\dot{P} &= [\, P, \Omega_\rho] \,, \\
\Omega_\rho &= \frac{1}{2}\frac{\partial \mathfrak{h}_{m-2-\rho}}{\partial Z} = \{Z, B^\rho\} + \{P, B^{\rho+1}\} \in so(m).
\end{aligned} \qquad (3.3)
$$

On the other hand, consider the following dynamical system on the variety $\mathcal{T}^r$

$$
\begin{aligned}
\dot{\mathcal{X}}^T &= -\Omega_\rho \mathcal{X}^T + P\mathcal{X}^T \mathcal{X}B^{\rho+1}\mathcal{X}^T \equiv -\Omega_\rho \mathcal{X}^T - \mathcal{V}^T \mathcal{X} B^{\rho+1} \mathcal{X}^T, \\
\dot{\mathcal{V}}^T &= -\Omega_\rho \mathcal{V}^T + P\mathcal{V}^T \, \mathcal{X}B^{\rho+1}\mathcal{X}^T \equiv -\Omega_\rho \mathcal{V}^T + \mathcal{X}^T \mathcal{V}\mathcal{V}^T \mathcal{X} B^{\rho+1} \mathcal{X}^T, \\
\dot{\mathcal{Y}}^T &= -\Omega_\rho \mathcal{Y}^T + \mathcal{Y}^T \mathcal{X} B^{\rho+1} \mathcal{V}^T + P B^{\rho+1} \mathcal{Y}^T + \mathcal{X}^T \Xi_\rho \\
&\equiv -\Omega_\rho \mathcal{Y}^T + \mathcal{Y}^T \mathcal{X} B^{\rho+1} \mathcal{V}^T - \mathcal{V}^T \mathcal{X} B^{\rho+1} \mathcal{Y}^T + \mathcal{X}^T \mathcal{V} B^{\rho+1} \mathcal{Y}^T + \mathcal{X}^T \Xi_\rho,
\end{aligned} \qquad (3.4)
$$

where Ξ_ρ is the $r/2 \times r/2$ symmetric matrix

$$
\begin{gathered}
\Xi_\rho = \mathcal{V}B^{\rho+1}\mathcal{Y}^T + \mathcal{Y}B^{\rho+1}\mathcal{V}^T - \mathcal{V}B^{\rho+2}\mathcal{V}^T + \frac{1}{2}(\Lambda + \Lambda^T), \\
\Lambda = \mathcal{X}B^{\rho+1}\mathcal{X}^T[\mathcal{V}\mathcal{Y}^T + \mathcal{Y}\mathcal{V}^T + \mathcal{V}B\mathcal{V}^T] + \mathcal{X}B^{\rho+2}\mathcal{X}^T\mathcal{V}\mathcal{V}^T - \mathcal{X}B\mathcal{X}^T\mathcal{V}\mathcal{V}^T\mathcal{X}B^{\rho+1}\mathcal{X}^T,
\end{gathered}
$$

and where one must substitute the above expression for Ω_ρ and then the expressions (3.1).

The matrices Ξ_ρ are chosen in such a way that equations (3.4) preserve the constraints (3.2) and therefore indeed describe a flow on $\mathcal{T}^r$.

Theorem 3.1 *1). Under the substitution* (3.1) *solutions of the system* (3.4) *pass to rank r solutions of the multidimensional Steklov system (3.3).*

2). Up to the action of the discrete group generated by reflections $(\mathcal{X},\mathcal{Y},\mathcal{V}) \to (-\mathcal{X},-\mathcal{Y},-\mathcal{V})$ *the system* (3.4) *is described by the following Lax pair with* $r\times r$ *matrices and rational parameter* λ

$$
\dot L(\lambda) = [L(\lambda), A_\rho(\lambda)], \tag{3.5}
$$

$$
\begin{aligned}
L(\lambda) &= \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I}-B)^{-1}[\mathcal{Y}+\lambda\mathcal{V}]^T & \mathcal{X}(\lambda\mathbf{I}-B)^{-1}\mathcal{X}^T \\ -(\mathcal{Y}+\lambda\mathcal{V})(\lambda\mathbf{I}-B)^{-1}[\mathcal{Y}+\lambda\mathcal{V}]^T & -[\mathcal{Y}+\lambda\mathcal{V}](\lambda\mathbf{I}-B)^{-1}\mathcal{X}^T \end{pmatrix} \\
&= L_1\lambda + L_0 + \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I}-B)^{-1}\mathcal{Z}^T & \mathcal{X}(\lambda\mathbf{I}-B)^{-1}\mathcal{X}^T \\ -\mathcal{Z}(\lambda\mathbf{I}-B)^{-1}\mathcal{Z}^T & -\mathcal{Z}(\lambda\mathbf{I}-B)^{-1}\mathcal{X}^T \end{pmatrix},
\end{aligned} \tag{3.6}
$$

where $\mathcal{Z}=\mathcal{Y}+B\mathcal{V}$,

$$
L_1 = \begin{pmatrix} 0 & 0 \\ -\mathcal{V}\mathcal{V}^T & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} \mathcal{X}\mathcal{V}^T & 0 \\ -\mathcal{V}B\mathcal{V}^T - \mathcal{V}\mathcal{Y}^T - \mathcal{Y}\mathcal{V}^T & -\mathcal{V}\mathcal{X}^T \end{pmatrix},
$$

and

$$
A(\lambda)_\rho = \begin{pmatrix} \mathcal{X}\,\mathcal{B}(\lambda)\,[\mathcal{Y}+\mathcal{V}B]^T & \mathcal{X}\,\mathcal{B}(\lambda)\,\mathcal{X}^T \\ -[\mathcal{Y}+\lambda\mathcal{V}]\,\mathcal{B}(\lambda)\,[\mathcal{Y}+\lambda\mathcal{V}]^T + & -[\mathcal{Y}+\mathcal{V}B]\,\mathcal{B}(\lambda)\,\mathcal{X}^T \end{pmatrix},
$$

$$
\mathcal{B}(\lambda) = \lambda^\rho\mathbf{I} + \lambda^{\rho-1}B + \cdots + B^\rho .
$$

3). The coefficients of the characteristic polynomial $|\Phi(\lambda)\,L(\lambda) - w\mathbf{I}|$ *are functions of the right hand sides of* (3.1)*, and they can be expressed only in terms of* Z_{ij}, P_{ij} *as follows*

$$
|w\mathbf{I} - \Phi(\lambda)\,L(\lambda)| = w^r + \sum_l w^{r-l}\Phi^{l-1}(\lambda)\,\tilde{\mathcal{I}}_l(\lambda, Z, P), \qquad l = 2,4,\ldots,r,
$$

$$
\Phi(\lambda) = (\lambda - b_1)\cdots(\lambda - b_m),
$$

thus giving all nonzero invariant polynomials $\tilde{\mathcal{I}}_2(\lambda),\ldots,\tilde{\mathcal{I}}_r(\lambda)$ *on* $\mathcal{W}^r$.

In view of Theorem 3.1 one can say that (3.5) *is a Lax representation with a rational parameter for multidimensional Steklov–Lyapunov systems restricted onto* $\mathcal{W}^r \subset so(m)\times so(m)$. Notice that, according to item 3, for $Z,P\in\mathcal{W}^r$, the spectral curve of the hyperelliptic Lax pair (2.5) is birationally equivalent to that of the rational Lax pair (3.5).

Proof of Theorem 3.1. 1). Differentiating left and right hand sides of (3.1) by virtue of equations (3.3) and (3.4) respectively, we find that both derivatives coincide under the substitution (3.1).

2). We differentiate $L(\lambda)$ along the flow of the system (3.4). In view of matrix relations in (3.2) and the identity $(\lambda\mathbf{I}-B)^{-1}B = \lambda(\lambda\mathbf{I}-B)^{-1}-\mathbf{I}$, the result coincides with the commutator in (3.5).

3). First, notice that $L(\lambda) \in \mathrm{sp}\,(r/2)$, hence all the odd-order diagonal minors of $L(\lambda)$ equal zero. The sum of all the diagonal minors of even order k of $\Phi(\lambda)L(\lambda)$ can be represented in the form

$$\Phi^k(\lambda) \sum_I \frac{1}{(\lambda - b_{i_1}) \dots (\lambda - b_{i_k})} \left(\sum \mathcal{M}_{i_1 i_2} \cdots \mathcal{M}_{i_{k-1} i_k} \right)^2,$$

where

$$\mathcal{M}_{ij} = \sum_{s=1}^{r/2} \left(x_i^{(s)} (y_j^{(s)} + \lambda v_j^{(s)}) - x_j^{(s)} (y_i^{(s)} + \lambda v_i^{(s)}) \right),$$
$$\{i_1 < \cdots < i_k\} \subset \{1, \dots, m\},$$

which, in view of (3.1) and (2.13), coincides with the polynomial $\Phi^{k-1}(\lambda)\,\tilde{\mathcal{I}}_k(\lambda)$. This establishes item 3 of Theorem 3.1. $\square$

4 Reductions in the rank 2 case

Now we consider in detail the simplest case of the motion on rank 2 coadjoint orbits $\mathcal{S}^2_{c,d} \subset so(m) \times s(m)$, which nevertheless are generic in the classical problem $m = 3$. As follows from Proposition 2.1, these orbits are $4(m-2)$-dimensional and on them all the invariant polynomials $\widetilde{\mathcal{I}}_6(s, Z, P), \dots, \widetilde{\mathcal{I}}_g(s, Z, P)$ and two leading coefficients of $\widetilde{\mathcal{I}}_4(s, Z, P)$ (Casimir functions) are identically zero. According to (2.18), the set of $2(m-2)$ nonzero independent integrals and two quadratic Casimir functions is given by the coefficients of the polynomials

$$\begin{aligned} \widetilde{\mathcal{I}}_2(s, Z, P) &= \sum_{1 \le i < j \le m} \frac{\Phi(s)}{(s - b_i)(s - b_j)} (Z + sP)^2 = \sum_{\mu=0}^{m} s^\mu H_{2\mu}(Z, P), \\ \widetilde{\mathcal{I}}_4(s, Z) &= \sum_I \frac{\Phi(s)}{(s - b_{i_1}) \dots (s - b_{i_4})} |Z|^I_I = \sum_{\mu=0}^{m-4} s^\mu H_{4\mu}(Z), \end{aligned} \tag{4.1}$$

where now $I = \{i_1 \cdots i_4\}$, $i_1 < \cdots < i_4$. The subvariety

$$\mathcal{F}^2_{c,d} = \{(Z, P) \in \mathcal{S}^2_{c,d} \mid \forall s \in \mathbb{R},\ \operatorname{rank}|Z + sP| = 2\}$$

is obtained by fixing to zero $m - 3$ quartic Hamiltonians $H_{4,0}, \dots, H_{4,m-4}$. Thus $\mathcal{F}^2_{c,d}$ has dimension $3m - 5$. Equivalently, $\mathcal{F}^2_{c,d}$ can be defined as the intersection of the orbit $\mathcal{S}^2_{c,d}$ with the quadrics

$$\left\{ Pf(|Z|^I_I) \equiv Z_{i_1 i_2} Z_{i_3 i_4} - Z_{i_1 i_3} Z_{i_4 i_2} + Z_{i_2 i_3} Z_{i_1 i_4} = 0 \right\},$$

$Pf(|Z|^I_I)$ being the Pfaffian of the 4×4 determinant $|Z|^I_I$.

On $\mathcal{F}^2_{c,d}$ and on $\mathcal{W}^2$ the flows generated by the quartic Hamiltonians $H_{4,0}, \dots, H_{4,m-4}$ are zero. Instead, we consider the flows of the *quadratic* Hamiltonians $\mathcal{H}_I = Pf(|Z|^I_I)$, which, in view of equations (2.2), have the simple matrix form

$$Z' = B\hat{Z}_I P - P\hat{Z}_I B, \quad P' = P\hat{Z}_I - \hat{Z}_I P, \qquad (\hat{Z}_I)_{ij} = \frac{\partial Pf\,(|Z|^I_I)}{\partial Z_{ij}}. \tag{4.2}$$

One can show by hand that for any 4-indices I, J, the Poisson bracket $\{\mathcal{H}_I, \mathcal{H}_J\}_1$ is a linear combination of the functions $\mathcal{H}_I$, hence on $\mathcal{W}^2$ they commute with each other. As we shall see later (item 3 of Theorem 4.6), all $\mathcal{H}_I$ also commute with the coefficients of $\widetilde{\mathcal{I}}_2(s, Z, P)$.

Notice that the corresponding flows (4.2) do not commute even on $\mathcal{W}^2$! In the sequel we denote these flows by $\mathcal{P}_I$.

Special Poisson Reduction. Below we are going to make a kind of reduction with respect to the flows $\mathcal{P}_I$, which is similar to the classical Marsden–Weinstein reduction by an action of a finite-dimensional Lie group ([14]). However, in our case there is no action and, moreover, the integrals of the system into consideration $(\mathcal{H}_I)$ are not general, but partial. That is why we want now to describe briefly our reduction procedure from a more abstract point of view.

Theorem 4.1 *Suppose we have a Hamiltonian system $\dot{x} = X_H(x)$ with a Hamiltonian $H(x)$ on a symplectic manifold $\mathcal{M}$ with local coordinates x and the Poisson bracket $\{\cdot,\cdot\}_*$, and there are k functions $f_l(x)$ satisfying the following properties:*

1) the common level $\mathcal{M}_0 = \{f_1 = 0, \dots, f_k = 0\}$ is a smooth submanifold of codimension k; in particular, the differentials of these functions are linearly independent on $\mathcal{M}_0$,

2) the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_k}$ are all tangent to $\mathcal{M}_0$ or, which is the same, $\{f_i, f_j\}_* = 0$ on $\mathcal{M}_0$,

3) $\{f_i, H\}_* = 0$ on $\mathcal{M}_0$ for every $i = 1, \dots, k$.

Then

1) The distribution on $\mathcal{M}_0$ generated by $X_{f_1}, \dots, X_{f_k}$ is integrable and, therefore, it forms a foliation ζ on $\mathcal{M}_0$ of dimension k.

2) For the case of compact leaves of the foliation, the quotient space $\mathcal{M}_0/\zeta$ (obtained by identifying each leaf of ξ into a point) has a natural symplectic structure and the initial Hamiltonian system $\dot{x} = X_H(x)$ can naturally be reduced onto $\mathcal{M}_0/\zeta$.

We note that the space $\mathcal{M}_0/\zeta$ may have singular points. In this case the reduced symplectic structure exists only on the regular open subset of $\mathcal{M}_0/\zeta$.

The precise description of the symplectic structure on $\mathcal{M}_0/\zeta$ is given in terms of a reduced Poisson bracket on the quotient space as follows. Let g, h be two arbitrary smooth functions on $\mathcal{M}_0/\zeta$. These functions are naturally identified with functions $\tilde{g}$, $\tilde{h}$ on $\mathcal{M}_0$ which are constant on the leaves of ζ.

To define the reduced bracket $\{g, h\}$ we simply want to take the Poisson bracket of $\tilde{g}$ and $\tilde{h}$. But to do so we need to extend $\tilde{g}$ and $\tilde{h}$ from $\mathcal{M}_0$ to the whole $\mathcal{M}$, because there is no natural Poisson structure on $\mathcal{M}_0$. Let $\hat{g}, \hat{h}$ be any smooth functions on $\mathcal{M}$ such that $\tilde{g} = \hat{g}|_{\mathcal{M}_0}$, $\tilde{h} = \hat{h}|_{\mathcal{M}_0}$.

Proposition 4.2 *1) The restriction of $\{\hat{g}, \hat{h}\}_*$ onto $\mathcal{M}_0$ does not depend on the choice of $\hat{g}$ and $\hat{h}$;*

2) $\{\hat{g}, \hat{h}\}_*|_{\mathcal{M}_0}$ is a first integral of the Hamiltonian flows $X_{f_1}, \dots, X_{f_k}$, i.e., it is constant on the leaves of the foliation ζ and, therefore, can be regarded as a function on the reduced space $\mathcal{M}_0/\zeta$.

The function so obtained is, by definition, the (reduced) Poisson bracket $\{g, h\}_{\mathrm{r}ed}$ on $\mathcal{M}_0/\zeta$. One can show that this bracket is non-degenerate almost everywhere, so $\mathcal{M}_0/\zeta$ obtains a natural symplectic structure. The proof goes along the same lines as in [14]. Since the original Hamiltonian H is invariant with respect to X_{f_i} (condition 3 of the theorem), the reduced Hamiltonian on $\mathcal{M}_0/\zeta$ and the corresponding reduced Hamiltonian system are correctly defined.

Proof of Theorem 4.1. Since $\{f_i, f_j\}_* \equiv 0$ on $\mathcal{M}_0$, the differential of the bracket $\{f_i, f_j\}_*$ considered as a function on $\mathcal{M}$ is a linear combination of $df_1(x), \dots, df_k(x)$ at each point $x \in \mathcal{M}_0$. Hence, for $x \in \mathcal{M}_0$ we have

$$\begin{aligned} [X_{f_i}, X_{f_j}](x) &= -X_{\{f_i, f_j\}_*}(x) = -\omega^{-1}(d\{f_i, f_j\}_*(x)) \\ &= -\omega^{-1}(\sum_{l=1}^{k} c_{ijl}(x) df_l(x)) = \sum_{l=1}^{k} c_{ijl}(x) X_{f_l}(x)\,, \end{aligned}$$

$c_{ijl}(x)$ being certain functions. Notice that this relation takes place only on $\mathcal{M}_0$ and nowhere else in general. Thus the Frobenius integrability condition holds, which establishes item 1). Item 2) follows from Proposition 4.2.

Proof of Proposition 4.2. 1) Let $\hat{g}$, $\hat{g}'$ be two different functions both satisfying $\hat{g}|_{\mathcal{M}_0} = \tilde{g}$, $\hat{g}'|_{\mathcal{M}_0} = \tilde{g}$. To show that $\{\hat{g}, \hat{h}\}_*|_{\mathcal{M}_0} = \{\hat{g}', \hat{h}\}_*|_{\mathcal{M}_0}$ it suffices to verify that $\{\hat{g} - \hat{g}', \hat{h}\}_*|_{\mathcal{M}_0} = 0$. We now use the fact that the function $\hat{g} - \hat{g}'$ is identically zero on $\mathcal{M}_0$. This implies that at each point $x \in \mathcal{M}_0$, $d(\hat{g} - \hat{g}')$ is a linear combination of $d\,f_1, \dots, d\,f_k$. Hence,

$$\begin{aligned} \{\hat{g} - \hat{g}', \hat{h}\}_*(x) = -\langle d(\hat{g} - \hat{g}')(x), X_{\hat{h}}(x)\rangle = -\left\langle \sum_{l=1}^{k} c_l(x) df_l(x), X_{\hat{h}}(x) \right\rangle = \\ = -\sum_{l=1}^{k} C_l(x)\langle df_l(x), X_{\hat{h}}(x)\rangle = -\sum_{l=1}^{k} C_l(x)\{f_l, \hat{h}\}_*(x), \end{aligned}$$

$C_l(x)$ being certain functions. Now, since $\{f_l, \hat{h}\}_*|_{\mathcal{M}_0} = 0$ for any $1 \le l \le k$, we obtain the required result.

2) It remains to show that the function $\{\hat{g}, \hat{h}\}_*|_{\mathcal{M}_0}$ is invariant under the flows $X_{f_1}, \dots, X_{f_k}$. This is equivalent to conditions $\{f_i, \{\hat{g}, \hat{h}\}\}_*|_{\mathcal{M}_0} \equiv 0$. We have

$$\{f_i, \{\hat{g}, \hat{h}\}_*\}_* = -\{\hat{g}, \{\hat{h}, f_i\}_*\}_* + \{\hat{h}, \{\hat{g}, f_i\}_*\}_*.$$

Since $\{\hat{h}, f_i\}_*|_{\mathcal{M}_0} \equiv 0$ and $\{\hat{g}, f_i\}_*|_{\mathcal{M}_0} \equiv 0$, we arrive at item 2).

In the above construction we assumed the functions $f_1, \dots, f_k$ to be independent on $\mathcal{M}_0$. However, everything can be repeated under the weaker assumption that the submanifold $\mathcal{M}_0$ is coisotropic or, which is the same, $\operatorname{codim} \mathcal{M}_0 = \operatorname{corank}(\omega|_{T\mathcal{M}_0})$.

Below we apply this construction in our case. As the symplectic manifold $\mathcal{M}$ and its submanifold $\mathcal{M}_0$ we shall consider the rank 2 orbit $S^2_{c,d}$ and the common level surface of the Pfaffians $Pf(|Z|)^I_I$ respectively.

Steklov–Lyapunov flows and the flows $\mathcal{P}_I$ on $\mathcal{T}^2$. In the rank 2 case, the matrices $\mathcal{X}^T, \mathcal{Y}^T, \mathcal{V}^T$ in relations (3.1) become just vectors x, y, v, whereas the relations themselves take the form

$$Z = x \wedge y, \quad P = x \wedge v, \qquad x, y, v \in \mathbb{R}^m\,. \tag{4.3}$$

The constraint submanifold $\mathcal{T}^2 \in \mathbb{R}^{3m}$ is defined by three conditions

$$(x, x) = 1, \quad (x, v) = 0, \quad (x, y + Bv) = 0. \tag{4.4}$$

Notice that in view of (4.3), $x_1, \dots, x_m$ become homogeneous coordinates of the focus of pencil of lines $\mathcal{L} = \{Z + sP\}$ in $\mathbb{P}^n$ $(n = m - 1)$.

The formulas (4.3) can be inverted to give a pair of points on $\mathcal{T}^2$ in view of the following proposition.

Proposition 4.3 *Let $\mathfrak{w}_{123} \subset \mathcal{W}^2$ be a domain defined by the conditions $Z_{\alpha\beta} = P_{\alpha\beta} = 0$, $\alpha, \beta = 1, 2, 3$. Then the redundant coordinates x, y, v can be expressed in terms of Z, P on the open subset $\mathcal{W}^2 \setminus \mathfrak{w}_{123}$ as follows*

$$x = \pm \bar{x}^{(123)} / \left| \bar{x}^{(123)} \right|, \quad v = -Px, \quad y = (-Z + (x, BPx))\, x, \tag{4.5}$$

where

$$\begin{aligned}
\bar{x}_1^{(123)} &= Z_{12}P_{13} - Z_{13}P_{12}\,, \\
\bar{x}_2^{(123)} &= Z_{23}P_{21} - Z_{21}P_{23}\,, \\
\bar{x}_3^{(123)} &= Z_{31}P_{32} - Z_{32}P_{31}\,, \\
\bar{x}_j^{(123)} &= -(Z_{12}P_{3j} - Z_{13}P_{2j} + Z_{23}P_{1j})\,, \qquad j = 4, \dots, m\,.
\end{aligned} \tag{4.6}$$

Expressions on other open subsets $\mathcal{W}^2 \setminus \mathfrak{w}_{\alpha\beta\gamma}$ are obtained from (4.6) by the corresponding permutation of indices.

Note that in the classical case $m = 3$, in the vector variables (2.10) the above expressions take the form

$$
\begin{aligned}
x &= \frac{1}{\gamma} z \times p, \quad v = x \times p = \frac{1}{\gamma}[(p,z)p - (p,p)z], \\
y &= x \times z - (Bx, x \times p)x = \frac{1}{\gamma}[(z,z)p - (z,p)z] \\
&\quad - \frac{1}{\gamma^3}\Big(B(z \times p), (p,z)p - (p,p)z \Big)\, z \times p, \qquad \gamma = |z \times p|.
\end{aligned} \tag{4.7}
$$

Relations (4.3) and (4.5), (4.6) establish a two-to-one correspondence between$\mathcal{T}^2$ and $\mathcal{W}^2$: the triples x, y, v and $-x, -y, -v$ are mapped to the same pair Z, P.

Proof of Proposition 4.3. The formulas (4.6) can be checked by direct calculations. Their geometric proof is the following. Let $(X_1 : \cdots : X_m)$ be homogeneous coordinates in the projective space $\mathbb{P}^{m-1}$ and $Y_2 = X_2/X_1, \ldots, Y_m = X_m/X_1$ be Cartesian coordinates in $\mathbb{C}^{m-1} = \mathbb{P}^{m-1} \backslash \{X_1 = 0\}$. Now let $\ell_1, \ell_2 \subset \mathbb{P}^{m-1}$ be lines with Plücker coordinates Z_{ij}, P_{ij} respectively. Then their affine parts in $\mathbb{C}^{m-1}$ can be described in parametric form

$$
\begin{aligned}
&\Big\{ Y_i(\tau) = Z_{i1}\tau + \sum_{k=2}^{m} Z_{ik} Z_{k1} \,|\tau \in \mathbb{C} \Big\}, \quad \text{respectively} \\
&\Big\{ Y_i(\tau') = P_{i1}\tau' + \sum_{k=2}^{m} P_{ik} P_{k1} \,|\tau' \in \mathbb{C} \Big\}, \quad i = 2, \ldots, m.
\end{aligned} \tag{4.8}
$$

Without loss of generality, here we assume that $\sum_{i=2}^{m} Z_{1i}^2 = \sum_{i=2}^{m} P_{1i}^2 = 1$. According to the condition rank $|Z + sP| = 2$, the two lines intersect at a point (the focus of the pencil $\mathcal{L}$) , whose homogeneous coordinates in $\mathbb{P}^{m-1}$ give the components of x up to a common factor. Matching the right hand sides of the expressions in (4.8) and using the above normalization conditions, we find the values of τ, τ' corresponding to the intersection point, and, after some calculations, the expressions (4.6).

The formulas (4.5) are then obtained by applying the second and third conditions in (4.4). The proposition is proved. □

It appears that the flows $\mathcal{P}_I$ on $\mathcal{W}^2$ generated by the quadratic Hamiltonians $\mathcal{H}_I = Pf(|Z|_I^I)$ do not change the focus of the pencil of lines $\mathcal{L}$.

Proposition 4.4 *In vector variables x, y, v on $\mathcal{T}^2$ the flows (4.2) have the form*

$$
x' = 0, \quad v' = -\hat{Z}_I v, \quad y' = B\hat{Z}_I v, \qquad (\hat{Z}_I)_{ij} = \frac{\partial Pf(|Z|_I^I)}{\partial Z_{ij}}, \tag{4.9}
$$

where one must substitute $Z = x \wedge y$.

One can check that these flows preserve the constraints (4.4) and therefore are indeed flows on $\mathcal{T}^2$.

Sketch of a proof of Proposition 4.4. First, note that the condition rank $|Z+sP|=2$ for any $s\in\mathbb{R}$ implies

$$\underset{\varkappa=0}{Res}\, Pf(|Z+\varkappa^{-1}P|_I^I) \equiv Z_{i_1i_2}P_{i_3i_4} - Z_{i_1i_3}P_{i_4i_2} + Z_{i_2i_3}P_{i_1i_4} \\ + P_{i_1i_2}Z_{i_3i_4} - P_{i_1i_3}Z_{i_4i_2} + P_{i_2i_3}Z_{i_1i_4} = 0 \qquad (4.10)$$

for $i_1<i_2<i_3<i_4$. Calculating the derivatives of the homogeneous coordinates $\bar{x}_i$ in (4.6) with respect to any of the flows given by (4.2) and using the conditions $\mathcal{H}_I=0$ and (4.10), we find that the vector $\bar{x}'$ is a linear combination of alternative expressions for $\bar{x}$ obtained from the right hand sides of (4.6) by various permutations of indices. In particular,

$$\left\{\bar{x}_i^{(123)}, \mathcal{H}_{1234}\right\}_1 = (Z_{12}+b_3P_{12})\bar{x}_i^{(124)} + (Z_{13}+b_2P_{13})\bar{x}_i^{(134)} + (Z_{23}+b_1P_{23})\bar{x}_i^{(234)}, \\ i=1,\ldots,m.$$

This implies that $\bar{x}'$ is collinear to $\bar{x}$, hence the normalized vector x is constant.

Next, we substitute expressions (4.3) into the Hamilton equations (4.2) and take into account $x'=0$. As a result, comparing coefficients at different components of x_i, we arrive at two last equations in (4.9), which proves the proposition. □

Theorem 4.5 *The variables x_i commute with respect to the Poisson bracket (2.1), i.e., $\{x_i,x_j\}_1=0$.*

Proof. Since $\{x_i,\mathcal{H}_I\}_1=0$ for any i, from the Jacobi identity we have

$$\{\{x_i,x_j\}_1,\mathcal{H}_I\}_1 = -\{\{x_j,\mathcal{H}_I\}_1,x_i\} - \{\{\mathcal{H}_I,x_i\}_1,x_j\}_j = 0.$$

For $m=3$, when the flows $\mathcal{P}_I$ do not exist, the proof is direct. Namely, from the vector expressions (4.7) we have

$$\frac{\partial x_i}{\partial z_\alpha} = -\frac{x_\alpha}{\gamma}[p\times x]_i, \quad \frac{\partial x_i}{\partial p_\alpha} = \frac{x_\alpha}{\gamma}[z\times x]_i, \qquad \alpha=1,2,3. \qquad (4.11)$$

Substituting this into the vector analog of Hamiltonian equations (2.2) with $\mathcal{H}=x_i$, we obtain

$$z' = -\frac{[p\times x]_i}{\gamma}(z-Bp)\times x + \frac{[z\times x]_i}{\gamma}p\times x, \quad p' = -\frac{[p\times x]_i}{\gamma}p\times x,$$

where now prime denotes the derivative with respect to the flow with the Hamiltonian x_i. Hence, in view of (4.11),

$$\{x_i,x_j\}_1 = \left(\frac{\partial x_j}{\partial z}, z'\right) + \left(\frac{\partial x_j}{\partial p}, p'\right) = 0.$$

The theorem is proved.

The systems (3.4) on $\mathcal{T}^2$ take the form

$$\begin{aligned}
\dot{x} &= -\Omega_\rho x + (x, B^{\rho+1}x)\, Px, \\
\dot{v} &= -\Omega_\rho v + (x, B^{\rho+1}x)\, Pv, \\
\dot{y} &= -\Omega_\rho y + (x, B^{\rho+1}v)y - (x, B^{\rho+1}y)v + (y, B^{\rho+1}v)x + \chi_\rho x,
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
\Omega_\rho &= \{Z, B^\rho\} + \{P, B^{\rho+1}\}, \qquad P = x \wedge v, \quad Z = x \wedge y\,, \quad \rho \in \{0 \cup \mathbb{N}\}, \\
\chi_\rho &= 2(y,v)(x, B^{\rho+1}x) - 2(y, B^{\rho+1}v)(x,x) - (x,x)(v, B^{\rho+2}v) \\
&\quad +(v,v)(x, B^{\rho+2}x) - (x, B^{\rho+1}x)[(v,v)(x,Bx) - (x,x)(v,Bv)]\,,
\end{aligned}$$

and they admit 2×2 matrix Lax representation, which comes from (3.5),

$$\dot{L}(\lambda) = [L(\lambda), A_\rho(\lambda)], \qquad \lambda \in \mathbb{C}, \tag{4.13}$$

$$\begin{aligned}
L(\lambda) &= \sum_{i=1}^m \frac{1}{\lambda - b_i} \begin{pmatrix} x_i(y_i + \lambda v_i) & x_i^2 \\ -(y_i + \lambda v_i)^2 & -x_i(y_i + \lambda v_i) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -(v,v) & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ -(v,Bv) - 2(v,y) & 0 \end{pmatrix} \\
&\quad + \sum_{i=1}^m \frac{1}{\lambda - b_i} \begin{pmatrix} x_i(y_i + b_i v_i) & x_i^2 \\ -(y_i + b_i v_i)^2 & -x_i(y_i + b_i v_i) \end{pmatrix}\,,
\end{aligned} \tag{4.14}$$

$$A_\rho(\lambda) = \begin{pmatrix} (x, \mathcal{B}(\lambda)(y + Bv)) & (x, \mathcal{B}(\lambda)x) \\ -Q_\rho(\lambda) & -(x, \mathcal{B}(\lambda)(y + Bv)) \end{pmatrix}\,, \tag{4.15}$$

where, as above, $\mathcal{B}(\lambda) = \lambda^\rho \mathbf{I} + \lambda^{\rho-1}B + \cdots + B^\rho$ and $Q_\rho(\lambda)$ is a polynomial of degree $\rho+2$, whose coefficients are chosen uniquely from the condition $\frac{d}{dt}(x, y + Bv) = 0$.

In particular, in view of the constraints (4.4),

$$\begin{aligned}
A_0(\lambda) &= \begin{pmatrix} 0 & 1 \\ -Q_0(\lambda) & 0 \end{pmatrix} \quad A_1(\lambda) = \begin{pmatrix} (Bx, (y+Bv)) & \lambda + (x, Bx) \\ -Q_1(\lambda) & -(Bx, (y+Bv)) \end{pmatrix}, \\
Q_0 &= (v,v)\lambda^2 + [2(v,y) + (v,Bv) - (v,v)(x,Bx)]\,\lambda + (y,y) \\
&\quad + \Delta(v,Bv) - (x,Bx)[2(v,y) + (v,Bv) - (v,v)(x,Bx)] \\
&\quad - (v,v)[\Delta_2 + (x, B^2x)], \\
Q_1 &= (v,v)\lambda^3 + [2(v,y) + (v,Bv)]\,\lambda^2 + [(y,y) - 2\Delta(v,y) + 2(y,Bv) - (v,v)]\,\lambda
\end{aligned}$$

The spectral curve $\mathcal{C} = \{|\Phi(\lambda)\, L(\lambda) - wI| = 0\}$ is now an even order hyperelliptic curve of genus $g = m-1$, and under the substitution (4.3) it reads

$$\begin{aligned}
w^2 &= -\sum_{i<j}^m \frac{\Phi^2(\lambda)}{(\lambda - b_i)(\lambda - b_j)} [x_i(y_j + \lambda v_j) - x_j(y_i + \lambda v_i)]^2 \\
&= -\Phi(\lambda)\, \widetilde{\mathcal{I}}_2(\lambda, Z, P),
\end{aligned} \tag{4.16}$$

thus giving all the quadratic first integrals (2.16) of the Steklov–Lyapunov systems on $\mathcal{S}^2_{c,d}$ and on $\mathcal{W}^2$.

In view of the constraints (4.4), the polynomial Lax matrix $\hat{L}(\lambda) = \Phi(\lambda)L(\lambda)$ has the following structure

$$\hat{L}(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \tag{4.17}$$

$$U(\lambda) = \lambda^g + U_1\lambda^{g-1} + \cdots + U_g, \quad V(\lambda) = V_1\lambda^{g-1} + \cdots + V_{g+1},$$
$$W(\lambda) = -(v,v)\lambda^{g+2} - W_{-1}\lambda^{g+1} - W_0\lambda^g - \cdots - W_g, \qquad g = m-1.$$

The set of all such complex matrices forms a $3m$-dimensional linear space $\mathcal{E}_g$ spanned by the coefficients of the polynomials U, V, W. Following [15] (see also [2, 17]), $\mathcal{E}_g$ can be completed to the fiber bundle $\bar{\mathcal{E}}_g$ over the $(2g+2)$-dimensional base space spanned by the coefficients of the characteristic polynomial

$$R(\lambda) = -\det \hat{L}(\lambda) \equiv U(\lambda)W(\lambda) + V^2(\lambda)$$

and parameterizing the corresponding genus g hyperelliptic curves $\mathcal{C}$, with fibers being the Jacobian varieties of the curves.

As follows from (4.14), the Lax matrices constructed of the real vectors x, y, v form a $2m$-dimensional real subvariety $\mathcal{N}_m \subset \mathcal{E}_g$ specified by conditions $R(b_i) = 0$, $i = 1, \ldots, m$. In this case the two leading coefficients of $R(\lambda)$ are linear combinations of the quadratic Casimir functions $H_{2m}, H_{2,m-1}$ of the bracket (2.1).

It is seen that for $m > 3$, the dimension of $\mathcal{W}^2$ is bigger than that of $\mathcal{N}_m$, hence, in this case, the Lax pair (4.13) is not equivalent to equations (4.12).

Proposition 4.6 *1). The components of the Lax matrix $L(\lambda|x, y, v)$ in (4.14) are invariant with respect to the flows (4.9). Generic orbits of these flows in $\mathcal{T}^2$ are $(m-3)$-dimensional compact real algebraic varieties.*

2). $\mathcal{N}_m$ is the factor variety of $\mathcal{T}^2$ by the orbits of the flows and by the action of the discrete group $\mathfrak{R}$ generated by reflections $(x_i, y_i, v_i) \to (-x_i, -y_i, -v_i)$, $i = 1, \ldots, m$.

3). On $\mathcal{F}^2_{c,d}$ and $\mathcal{W}^2$ the Pfafians $Pf(|Z|)^I_I$ commute with the quadratic first integrals in (4.1).

4). Generic orbits of the flows $\mathcal{P}_I$ in $\mathcal{F}^2_{c,d}$ are $(m-3)$-dimensional real compact algebraic varieties.

Proof. First, notice that the flows (4.9) do not change the vectors $y + Bv$, which form the Laurent part of $L(\lambda)$ in (4.14). Next, we have $(v,v)' = 0$ and

$$(v, Bv)' + 2(v,y)' = -2(Bv, \hat{Z}_I v) - 2(y, \hat{Z}_I v) + 2(v, B\hat{Z}_I v) \equiv 2\langle y \wedge v, \widehat{(x \wedge y)}_I\rangle,$$

which is zero due to the definition of $\hat{Z}_I$ in (4.2).

Hence, the components of $L(\lambda)$ provide $2m$ *independent* algebraic first integrals of the flows, and, therefore, their orbits are algebraic varieties of dimension $\dim\mathcal{T}^2 - 2m = m-3$.

Further, from equations (4.9) and the constraints (4.4) we find that for each fixed orbit, the vector v lies on the sphere S^{m-2} in $\mathbb{R}^{m-1} = \{v \mid (v,x)=0\}$. On the other hand, since on each orbit $y+Bv=d$, $d=$const and $(v,Bv)+2(v,y)=$const, the same vector belongs to the quadric $2(d,v)+(v,Bv)=$const. As a result, each orbit is diffeomorphic to a connected component of the intersection of two $(m-2)$-dimensional quadrics in $\mathbb{R}^{m-1}$, which is a compact variety. This implies items 1.

Next, the components of $L(\lambda)$ are invariant with respect to reflections of $\mathfrak{R}$, which yields item 2.

Since the above flows preserve $L(\lambda)$, the corresponding flows $\mathcal{P}_I$ on $\mathcal{W}^2$ preserve the quadratic integrals in (4.1) as coefficients of the spectral curve of $L(\lambda)$. Thus, these integrals and $Pf(|Z|)_I^I$ commute on $\mathcal{W}^2$.

Item 4 is a reformulation of item 1 in terms of the coordinates Z, P on $\mathcal{W}^2$. □

Now let $\mathcal{O}^2_{c,d}$ be a $2(m-1)$-dimensional subvariety of $\mathcal{N}_m$ obtained by fixing the two leading coefficients in the polynomial (4.16), i.e., by fixing the two quadratic Casimir functions on $\mathcal{W}^2$. In view of item 2 of the above proposition, $\mathcal{O}^2_{c,d}$ can also be regarded as the factor variety of $\mathcal{F}^2_{c,d}$ by the orbits of the flows $\mathcal{P}_I$ and by the action of the discrete group $\mathfrak{R}'$ induced by $\mathfrak{R}$ on $\mathcal{F}^2_{c,d}$. In view of item 3), the quadratic integrals $H_{m-2}(Z,P),\dots,H_0(Z)$ in (4.1) are reducible to functions on $\mathcal{O}^2_{c,d}$.

Now we are in position to apply the special Poisson reduction described in Theorem 4.1.

Theorem 4.7 *1) The reduced manifold $\mathcal{O}^2_{c,d}$ has a natural nondegenerate Poisson structure, which is inherited from the bracket $\{\,,\,\}_1$ on $\mathcal{S}^2_{c,d}$ as a result of the special Marsden–Weinstein reduction procedure.*

The restrictions of the Steklov–Lyapunov systems with the quadratic Hamiltonians $H_k(Z,P)$ onto $\mathcal{F}^2_{c,d}$ are reduced to Hamiltonian systems on $\mathcal{O}^2_{c,d}$.

2) Generic invariant manifolds of the latter systems are $(m-1)$-dimensional tori, which are real parts of the Jacobian varieties of the hyperelliptic curves (4.16).

Proof. 1). Indeed, $\mathcal{S}^2_{c,d}$, $\mathcal{F}^2_{c,d}$, and $Pf(|Z|_I^I)$ can be identified with the manifolds $\mathcal{M}$, $\mathcal{M}_0$ and the functions f_l of Theorem 4.1 respectively. Then all the conditions of this theorem are satisfied and the reduced manifold $\mathcal{O}^2_{c,d}$ obtains a nondegenerate Poisson structure described in Proposition 4.2.

2).

As the symplectic manifold $\mathcal{M}$ and its submanifold L we shall consider the rank 2 orbit $S^2_{c,d}$ and the common level surface of the Pfaffians $Pf(|Z|)_I^I$ respectively.

as a special Poisson (Marsden–Weinstein) reduction of obtained by fixing the Hamiltonians to zero and factorizing by the action of the Hamiltonian flows (4.2) and by the group $\mathfrak{R}'$ action

To get a global view on the above manifolds, we represent them in the following commutative diagram where arrows denote the corresponding maps (embeddings or factorizations), and the map $\Lambda : \mathcal{W}^2 \to \mathcal{N}_m$ is given by the composition of the formulas of Proposition 4.3 and (4.14).

$$
\begin{tikzcd}
so(m) \times so(m) & \mathcal{W}^2 \arrow[l, "\tilde{\mathcal{I}}_4(s)=\cdots=\tilde{\mathcal{I}}_g(s)=0"'] \arrow[r, "\Lambda"] & \mathcal{N}_m \\
\mathcal{S}^2_{c,d} \arrow[u, "\cup"] & \mathcal{F}^2_{c,d} \arrow[u, "\cup"] \arrow[l, "H_{4,\mu}(Z)=0"'] \arrow[r, "/\mathcal{P}_I\, /\mathfrak{R}'"] & \mathcal{O}^2_{c,d}. \arrow[u, "\cup"]
\end{tikzcd}
$$

In the classical case $m = 3$ the above diagram simplifies: the 6-dimensional variety $\mathcal{W}^2$ coincides with the product $so(3) \times so(3)$ itself, and 4-dimensional orbits $\mathcal{S}^2_{c,d}$ are coverings of $\mathcal{O}^2_{c,d}$. They are foliated with 2-dimensional tori, whose complexifications are coverings of the Jacobians of genus 2 hyperelliptic curves $\mathcal{C}$.

Note that another 2×2 matrix Lax pair for the classical Steklov system written in different coordinates related to an integrable geodesic flow on $SO(4)$ was found in [6].

5 Linearization of flows and separation of variables in the rank 2 case

Let $P_1 = (\lambda_1, w_1), \dots, P_g = (\lambda_g, w_g)$ be a divisor of $g = m-1$ points on the spectral curve $\mathcal{C}$, whose coordinates satisfy equations

$$U(\lambda_k) = 0, \quad w_k = V(\lambda_k).$$

Since $U(\lambda)$ and $V(\lambda)$ are polynomial of degree g and $g-1$ respectively, then

$$U = (\lambda - \lambda_1) \cdots (\lambda - \lambda_g), \quad V = \sum_{k=1}^{g} w_k \frac{\prod_{l \neq k}(\lambda - \lambda_l)}{\prod_{l \neq k}(\lambda_k - \lambda_l)} . \tag{5.1}$$

Now, taking residue of the Lax matrix (4.13) at $\lambda = b_i$, we obtain

$$
\begin{gathered}
x_i^2 = \frac{(b_i - \lambda_1) \cdots (b_i - \lambda_{m-1})}{\prod_{j \neq i}(b_i - b_j)} , \\
y_i + b_i v_i = x_i \sum_{k=1}^{g} \frac{w_k}{(b_i - \lambda_k) \prod_{s \neq k} (\lambda_k - \lambda_s)} , \\
i = 1, \dots, m,
\end{gathered} \tag{5.2}
$$

The first set of these expressions implies that $\lambda_1, \dots, \lambda_g$ are spheroconic coordinates on the unit sphere $\{(x, x) = 1\}$,

Now let us fix constants of motion by setting

$$\widetilde{\mathcal{I}}_2(\lambda, Z, P) = \psi(\lambda), \quad \psi(\lambda) = h_m \lambda^m + \cdots + h_1 \lambda + h_0, \qquad h_0, h_1, \dots, h_m = \text{const},$$

so that, due to (4.16), $w_k = \sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}$.

Theorem 5.1 *Let $Z(t), P(t)$ be a solution of the Steklov–Lyapunov system on $\mathcal{W}^2$ with the quadratic Hamiltonian*

$$H_f = \frac{1}{2}\left(f_m H_{2,m}(P) + f_{m-1} H_{2,m-1}(P,Z) + \cdots + f_0 H_{20}(Z,P)\right), \qquad (5.3)$$
$$f_0, \ldots, f_{m-2} = \mathrm{const}$$

and constants of motion $H_{2,m}(P) = h_m, \ldots, H_{20}(Z,P) = h_0$. Then the evolution of the points (λ_k, w_k) is given by the following standard Abel–Jacobi equations involving g holomorphic differentials on the curve $\mathcal{C}$,

$$\sum_{k=1}^{m-1} \frac{\lambda_k^r\, d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = d\phi_r, \qquad r = 0, 1, \ldots, m-2, \qquad (5.4)$$

where $d\phi_r = f_r\, dt$.

Recall that $H_{2,m}(P)$, $H_{2,m-1}(P,Z)$ are Casimir functions of the bracket $\{\,,\}_1$ and notice the corresponding constants f_{m-1}, f_m do not appear in the right hand sides of (5.4).

In particular, for the generalized Steklov and Lyapunov systems described by the Hamiltonians (2.4) the above equations take the form respectively

$$\begin{cases} \displaystyle\sum_{k=1}^{g} \frac{d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \\ \qquad \cdots \quad \cdots \quad \cdots \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-2} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = dt, \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-1} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \end{cases} \qquad \begin{cases} \displaystyle\sum_{k=1}^{g} \frac{d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \\ \qquad \cdots \quad \cdots \quad \cdots \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-2} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-1} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = dt. \end{cases}$$

Note that, for the classical case $m = 3$, the variables λ_1, λ_2 were first introduced and the quadratures (5) were obtained by F. Kötter in [12].

Proof of Theorem 5.1. As follows from the Lax equations (4.13) and expressions for $\hat{L}(\lambda)$ in (4.17), for the system with the Hamiltonian $\mathfrak{h}_{m-2-\rho}$,

$$\dot{U}(\lambda) = 2V(\lambda)[\lambda^\rho + \lambda^{\rho-1}(x, Bx) + \cdots + (x, B^\rho x)] - 2U(\lambda)(x, \mathcal{B}(\lambda)(y + Bv)),$$

Setting here $\lambda = \lambda_k$ and taking into account (5.1), we obtain

$$\dot{\lambda}_k \prod_{s\neq k}(\lambda_k - \lambda_s) = 2w_k[\lambda_k^\rho + \lambda_k^{\rho-1}(x, Bx) + \cdots + (x, B^\rho x)].$$

Then, according to relations (2.16), for the motion with the quadratic Hamiltonian

$$H_{2,m-2-\rho}(Z,P) = \sum_{s=0}^{\rho}(-1)^s \Delta_s \mathfrak{h}_{m-2-\rho+s},$$

we have

$$\frac{\dot{\lambda}_k}{2w_k}\prod_{s\neq k}(\lambda_k-\lambda_s)=\lambda_k^\rho+\lambda_k^{\rho-1}[(x,Bx)-\Delta_1]+\lambda_k^{\rho-2}[(x,B^2x)-\Delta_1(x,Bx)+\Delta_2]$$
$$+\cdots+\lambda_k^0[(x,B^\rho x)-\Delta_1(x,B^{\rho-1}x)+\cdots+(-1)^\rho\Delta_\rho]. \qquad (5.5)$$

Now applying relations (5.2) and the known Jacobi identities, we represent the right hand side in form

$$\lambda_k^\rho-\sigma_1\lambda_k^{\rho-1}+\cdots+(-1)^\rho\sigma_\rho\lambda_k^0,$$

where $\sigma_s=(-1)^sU_s$ is the elementary symmetric polynomial of $\lambda_1,\dots,\lambda_g$ of degree s and, as above, U_s is the coefficients of $U(\lambda)$. Again, in view of the Jacobi identities, for $0\le r\le g-1=m-2$ we have

$$\sum_{k=1}^{g}\lambda_k^r\frac{\lambda_k^\rho-\sigma_1\lambda_k^{\rho-1}+\cdots+(-1)^\rho\sigma_\rho\lambda_k^0}{\prod_{s\neq k}(\lambda_k-\lambda_s)}=\delta_{m-2-\rho,r}.$$

This, together with (5.5), implies that for the system with the Hamiltonian $H_{2,m-2-\rho}(Z,P)$ the evolution of λ-coordinates is given by equations

$$\sum_{k=1}^{g}\frac{\lambda_k^r\,d\lambda_k}{2w_k}=\delta_{m-2-\rho,r}\,dt, \qquad r=0,\dots,g-1.$$

By linearity, we conclude that for the motion with the generic Hamiltonian (5.3) this evolution is described by the system (5.4). □

Now introduce variables

$$\mu_k=\frac{w_k}{\Phi(\lambda_k)}=\frac{\sqrt{\lambda^m\langle P,P\rangle+H_{2,m-1}\lambda^{m-1}+\cdots+H_{2,0}}}{\sqrt{(\lambda-\lambda_1)\cdots(\lambda-\lambda_g)}}. \qquad (5.6)$$

Theorem 5.2 *On the $2g$-dimensional manifold $\mathcal{O}^2_{c,d}$ the variables $(\lambda_1,\mu_1),\dots,(\lambda_g,\mu_g)$ form a complete set Darboux coordinates with respect to the Lie–Poisson bracket (2.1) on $so(m)\times so(m)$, i.e.,*

$$\{\lambda_k,\lambda_s\}_1=\{\mu_k,\mu_s\}_1=0, \quad \{\lambda_k,\mu_s\}_1=\delta_{ks}, \qquad k,s=1,\dots,g.$$

As a corollary, we find that for $m=3$, the Kötter variables $\lambda_1,\mu_1,\lambda_2,\mu_2$ are Darboux coordinates on the orbits $S^2_{c,d}=\mathcal{O}^2_{c,d}$ with respect to the standard Lie–Poisson bracket on $e^*(3)$.

Proof of Theorem 5.2. As follows from Theorem 5.1,

$$\{\phi_\rho,H_{2,r}\}_1=\delta_{\rho r}, \qquad \rho,r=0,1,\dots,m-2,$$

where ϕ_ρ are angle type variables defined in a neighborhood of a generic invariant torus. Also, $\{H_{2,\rho},H_{2,r}\}_1=0$. Hence, the reduction of the corresponding symplectic structure on the orbit $\mathcal{S}^2_{c,d}$ onto $\mathcal{O}^2_{c,d}$ can locally be represented as

$$\omega=\sum_{r=0}^{m-2}d\phi_r\wedge dh_r+\sum_{0\le\rho<r\le m-2}C_{\rho r}\,d\phi_\rho\wedge d\phi_r$$

with some coefficients $C_{\rho r}$. Next, due to (5.4) and (5.6),

$$d\phi_r = \sum_{k=1}^{g} \frac{\partial \mu_k(\lambda_k, h)}{\partial h_r}\, d\lambda_k,$$

which implies

$$\begin{aligned}
\omega &= \sum_{k=1}^{g} d\lambda_k \wedge \left[\sum_{r=0}^{m-2} \frac{\partial \mu_k(\lambda_k, h)}{\partial h_r}\, dh_r \right] + \sum_{0\le \rho < r \le m-2} C_{\rho r}\, d\phi_\rho \wedge d\phi_r \\
&\equiv \sum_{k=1}^{g} d\lambda_k \wedge d\mu_k + \sum_{1\le k<s\le g} \tilde C_{ks}\, d\lambda_k \wedge d\lambda_s
\end{aligned}$$

with some coefficients $\tilde C_{ks}$. On the other hand, Theorem 4.5 says that $\{x_i, x_j\}_1 = 0$, which, together with the first relations in (5.2), implies $\{\lambda_k, \lambda_s\}_1 = 0$. As a result, in the expression for ω we have $\tilde C_{ks} = 0$, which proves the theorem.

6 Conclusion

In this paper we considered integrable Steklov–Lyapunov systems on rank r coadjoint orbits $\mathcal{S}^r_{c,d}$ in $so(m) \times so(m)$ and on their invariant subvarieties $\mathcal{F}^r_{c,d}$. We showed that the latter systems, written in terms of matrix triplets $\mathcal{X}, \mathcal{V}, \mathcal{Y}$, admit $r \times r$ matrix Lax representation in a generalized Gaudin form.

It would be interesting to find an appropriate generalization of the Weinstein–Aronzjan formula (1.3) to the case of Lax matrices (1.7) .

In the rank 2 case we described a Marsden–Weinstein reduction of $\mathcal{S}^2_{c,d}$ onto symplectic $2(m-1)$-dimensional manifolds $\mathcal{O}^2_{c,d}$, which is foliated with $(m-1)$-dimensional Jacobians of hyperelliptic spectral curves, and indicated Darboux coordinates with respect to the original Lie–Poisson structure on $so(m) \times so(m)$. For $m = 3$, these coordinates coincide with the mysterious separating variables used by Kötter in order to reduce the systems on $e^*(3)$ to Abel–Jacobi quadratures. They can be used to construct action-angle variables for the classical systems.

The properties of analogous reduction for arbitrary rank r are still not understood completely.

On the other hand, it appears that adding to the Lax matrix $L(\lambda)$ in (3.5) a constant $r \times r$ matrix Y allows a similar description of other generalizations of the Steklov–Lyapunov systems. For example, consider the following matrix "hybrid" system on the phase space $(Z, P, e^{(1)}, \dots, e^{(k)})$, $Z, P \in so^*(m)$, $e^{(1)}, \dots, e^{(k)} \in \mathbb{R}^m$, $k \le m$ (see also [9])

$$\begin{aligned}
\dot Z &= ZPB - BPZ + [\,\Gamma, B\,]\,, \\
\dot P &= [\,P, PB + BP\,] + [\,P, Z\,]\,, \\
\dot \Gamma &= [\,\Gamma, Z\,] + \Gamma PB - BP\Gamma\,,
\end{aligned} \tag{6.1}$$

$$\Gamma = \varepsilon(e^{(1)} \otimes e^{(1)} + \cdots + e^{(k)} \otimes e^{(k)})\,, \quad B = \mathrm{diag}(b_1, \dots, b_m)\,,$$

which for $\varepsilon \to 0$ is reduced to the generalized Lyapunov system (2.7) with $\rho = 0$, whereas for $P \to 0$ it becomes the simplest system of the Clebsch–Perelomov-Bogoyavlensky hierarchy on the dual to the semi-direct product Lie algebra $so(m)\times_s \underbrace{(\mathbb{R}^m \times \cdots \times \mathbb{R}^m)}_{k \text{ times}}$ ([4]), i.e.,

$$\dot{Z} = [\,\Gamma, B\,]\,, \qquad \dot{\Gamma} = [\,\Gamma, Z\,]\,.$$

This describes the motion of a spherically symmetrical top with the angular velocity Z in the field of the quadratic potential $\frac{1}{2}\left(e^{(1)}, Be^{(1)}\right) + \cdots + \frac{1}{2}\left(e^{(k)}, Be^{(k)}\right)$.

We mention without a proof that, for an even number r, $2k \le r \le m$, the system (6.1) has invariant manifolds $\tilde{\mathcal{W}}^r$ given by the conditions

$$\forall s \in \mathbb{R}\,, \quad \operatorname{rank} |Z + sP| = r, \quad \operatorname{rank} \begin{pmatrix} Z & e^{(1)} & \cdots & e^{(r)} \\ -(e^{(1)})^T & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -(e^{(r)})^T & 0 & \cdots & 0 \end{pmatrix} = r.$$

Then, similarly to (3.1), on $\tilde{\mathcal{W}}^r$ the variables $Z, P, e^{(1)}, \dots, e^{(k)}$ can be represented in terms of $r/2 \times m$ matrices $\mathcal{X}, \mathcal{Y}, \mathcal{V}$ as follows

$$Z = \mathcal{X}^T\mathcal{Y} - \mathcal{Y}^T\mathcal{X}\,, \quad P = \mathcal{X}^T\mathcal{V} - \mathcal{V}^T\mathcal{X}\,, \quad e^{(1)} = x^{(1)}, \dots, e^{(r)} = x^{(k)},$$

and the restriction of equations (6.1) onto $\tilde{\mathcal{W}}^r$ admits $r\times r$ matrix Lax representation

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$
$$L(\lambda) = Y + \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\,\mathcal{V}]^T & \mathcal{X}(\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \\ -(\mathcal{Y} + \lambda\mathcal{V})(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\mathcal{V}]^T & -[\mathcal{Y} + \lambda\mathcal{V}](\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \end{pmatrix}, \quad (6.2)$$

with certain polynomial matrix $A(\lambda)$ and the constant matrix Y of the following structure

$$Y = \begin{pmatrix} 0 & 0 \\ \mathbf{I}_k & 0 \end{pmatrix}, \qquad \mathbf{I}_k = \operatorname{diag}(\underbrace{1, \dots, 1}_{k \text{ units}}, 0, \dots, 0).$$

A detailed description of a natural Poisson structure on the space of Lax matrices (6.2) and its relation to symplectic properties of various Steklov–Lyapunov type systems, as well as their integrable discretizations, are left for a future publication.

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